Linear and Topological Properties of a Fiber Bundle to Solve Hodge Conjecture

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Abstract

On this work the topological manifold and its Kähler manifold of $\mathbb{C}$ was constructed. Hodge Conjectures was proved using an integration scheme to define cohomology and homology.

[1]

1 Hodge Conjecture.

On a projective non-singular algebraic variety over $\mathbb{C}$, any Hodge class is a rational linear combination of classes cl(Z) of algebraic cycles [6] .

2 Hypothesis

Def. Let X be a compact complex manifold of complex dimension n. Then X is an orientable smooth manifold of real dimension 2n, so its cohomology groups lie in degrees zero through 2n. Assume X is a Kähler manifold, so that there is a decomposition on its cohomology with complex coefficients

$$H^n(X, C) = \bigoplus_{p+q=n} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the subgroup of cohomology classes which are represented by harmonic functions of type (p, q). That is, these are the cohomology classes represented by differential forms which, in some choice of local coordinates $z_1, ..., z_n$ can be written as a harmonic function times

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$$dz_{i_1} \wedge ... \wedge dz_{z_{i_p}} \wedge d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_q}.$$

Since X is a compact oriented manifold, X has a fundamental class, and so X can be integrated over.

Let Z be a complex submanifold of X of dimension k, and let $i : Z \to X$ be the inclusion map. Choose a differential form α of type (p,q). We can integrate α over Z using the pullback function $i*$,

$$\int_Z i * \alpha$$

To evaluate this integral, choose a point of Z and call it $z = (z_1, ..., z_k)$. The inclusion of Z in X means that we can choose a local basis on X and have $z_{k+1} = ... = z_n = 0$ (rank-nullity theorem). If $p > k$, then α must contain some dz_i where z_i pulls back to zero on Z. The same is true for $d\bar{z}_j$ if $q > k$. Consequently, this integral is zero if $(p, q) \neq (k, k)$.

Let

$$Hdg^k(X) = H^{2k}(X, Q) \cap H^{k,k}(X)$$

we call this the group of *Hodge classes* of degree 2k on X

Hodge conjecture. Let X be a non-singular complex projective manifold. Then every Hodge class on X is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of X.

Which cohomology classes in $H^{k,k}(X)$ come from complex subvarieties Z?

Proof.

5

Def. Open Cover. Formally, a topological space X is called "compact" if every open cover of X has a finite subcover. That is, X is compact if for every collection C if open subsets of X such that

$$X = \bigcup_{S \in C} S$$

there is a **finite** subcollection $F \subseteq C$ such that

$$X = \bigcup_{S \in F} S$$

Def. A finite set is a set that has a finite number of elements. Informally, a finite set is a set which one could in principle count and finish counting.

Def. A topology on a set X may be defined as a collection τ of subsets of X, called **open sets** and satisfying the following axioms:

1. The empty set and X itself belong to τ

2. Any arbitrary (finite or infinite) union of members of τ belongs to τ

3. The intersection of any finite number of members of τ belongs to τ

Proof. With $(a, b) \in \mathbb{C}$ and $a,\ b \in \Re$ and $\sqrt{-z} =_A \sqrt{z}$ $(:= \sqrt{-z} < \sqrt{z})$ see ref[2, 3, 4, 5] .

Lets define an open set $B_r(a, b)$ on $\mathbb{C}$ like

$$B_r(a, b) = \{(x_1, x_2) \in \mathbb{C} \,|\, d[(a, b), (x_1, x_2)] < r\}$$

with

$d[(a, b), (x_1, x_2)] :=$ euclidean distance

1)

Now lets take $(x_1, x_2) \in \mathbb{C}$

If $(x_1, x_2) \in B_r(a, b)$ or $(x_1, x_2) \in B_r(c, d)$

Hence

$$(x_1, x_2) \in B_r(a, b) \cup B_r(c, d) = B_r \subseteq \mathbb{C}$$

Therefore

$$\bigcup_{B_r \in B_r} B_r(x_1, x_2) \in \tau$$

2) If $(x_1, x_2) \in B_r(a, b)$ and $(x_1, x_2) \in B_r(c, d)$

hence

$$(x_1, x_2) \in B_r(a, b) \cap B_r(c, d) = B_r \subseteq \mathbb{C}$$

hence

$$B_r(a, b) \cap B_r(c, d) \in \tau$$

3) If $B_r = \mathbb{C}$ hence $B_r \in \tau$

Therefore $\mathbb{C}$ is a topological space

4) Lets take circle equation

$$(x - a)^2 + (y - b)^2 = r^2$$

now with $b = 0$ (or $a = 0$ following same procedure for x_2) lets define next set

$$c = \{(x_1, x_2) \in \mathbb{C} \quad | \quad -(a + r) \leq x_1 \leq a + r; -\infty < x_2 < \infty; a = p\sqrt[n]{\pi^m}\}$$

with

$$p \in \{1, 2, 3, ...\}$$

with $B_r(x_1, x_2) \subset c$ an open set on τ

now consider that p is an index of $B_r(x_1, x_2)_p$ such that

$$\bigcup_{p \in 1,2,3,...} B_r(x_1, x_2)_p = \mathbb{C}$$

Hence $\bigcup_p B_r(x_1, x_2)_p$ is a cover of $\mathbb{C}$

Def. Subcover. Let C be a cover of a topological space X. A "subcover" of C is a subset of C that still covers X. The cover C is said to be an "open cover" if each of its members is an open set. That is, each U_α is contained in T, where T is the topology on X.

Proof.

Lets define next index

$c^* = \{(x_1, x_2) \in \mathbb{C} \quad | \quad -(a-r) \leq x_1 \leq a-r \quad ; -\infty < x_2 < \infty \quad ; a = p\sqrt[n]{\pi^m} \quad ; p \in \{1, 2, 3, ...\} \quad \}$

with $B_r(x_1, x_2) \subset c*$ an open set on τ

Hence

$$\bigcup_{p \in 1,2,3,...} B_r(x_1, x_2)_p = \mathbb{C}$$

Hence $c*$ is a subcover of $\mathbb{C}$

Like c* is indexed and $c* \subset c$

Therefore $\mathbb{C}$ is compact

Def. Complex manifold. In differential geometry and complex geometry, a complex manifold is a manifold with a "complex" structure, that is an atlas of charts to the open unit disc in the complex coordinate space C^n, such that the transition maps are holomorphic.

Proof

Def. A topological manifold of n dimension is a topological space M that must satisfy:

1. Be locally euclidean (i.e. for every point $x \in M$ exists an open U, neighborhood of x, homeomorphic by

$$\phi : U \to V$$

an open V of $\Re^n$)

Proof.

Lets consider next functions (see ref [2, 3, 4])

$$Add_{map} : \mathbb{C} \to \Re \quad ;$$

$$Add_{map}(x_1, x_2) = x_1 + x_2 \quad ;$$

$$Prod_{map} : \mathbb{C} \to \Re \quad ;$$

$$Prod_{map}(x_1, x_2) = (x_1)(x_2)$$

(see ref[2,3,4])

And lets define

$$f(x_1, x_2) := g(Prod_*(x_1, x_2))$$

and g an homomorphism

$$f(x_1, x_2) = g(Prod_*(x_1, x_2)) = Prod_\bullet(g(x_1), g(x_2));$$

$$\mathbb{C} \quad \to \quad \Re \quad \to \quad \Re$$

such that

Hyp.

$$g(Prod_*(x_1, x_2)) = g(x_1 * x_2) = g(x_1) \bullet g(x_2) = Prod_\bullet(g(x_1), g(x_2)) \in \Re$$

and $(x_1, x_2) \in \mathbb{C}$

1.1. f is *one - to - one*

Lets take (x_1, x_2) and (x_3, x_4)

if

$$f(x_1, x_2) = f(x_3, x_4)$$

hence

$$g(x_1 * x_2) = g(x_1) \bullet g(x_2) = g(x_3) \bullet g(x_4) = g(x_3 * x_4)$$

hence

$$x_1 * x_2 = x_3 * x_4$$

if $Prod_*(a, b) = a * b$ is a left or right product hence

$$x_1 = x_3 \quad ;$$

and

$$x_2 = x_4$$

Therefore f is one - to - one

1.2. f is *onto*

$$\forall y = g(x_1 * x_2) = g(x_1) \bullet g(x_2) \in Y = \Re; \quad \exists (x_1, x_2) \in \mathbb{C} \quad | \quad f(x_1, x_2) = g(Prod_*(x_1, x_2))$$

$$= g(x_1 * x_2) = g(x_1) \bullet g(x_2) = Prod_\bullet(g(x_1), g(x_2))$$

hence f is onto

1.3. f is continuous

Def. A continuous function $f : X \to Y$ is continuous if for every open set $V \subseteq Y$, the inverse image

$$f^{-1}(V) = \{x \in X | f(x) \in V\}$$

is an open subset of X

Proof.

Lets define an open subset on $\Re$

$$B_r(p) = \{x \in \Re \quad | \quad d(x,p) < r\}$$

now consider

$$g(Prod_*(x_1,x_2)) = g(x_1 * x_2) = g(x_1) \bullet g(x_2) = Prod_\bullet(g(x_1), g(x_2)) = f(x_1,x_2) \in \Re$$

Like $\Re$ is a topological space hence exist a $B_r(p)$ for each $f(x_1,x_2) \in \Re$; i.e.

$$f(x_1,x_2) \in B_r(p) \subseteq \Re$$

by onto property about f ; hence $\exists (x_1,x_2) \in \mathbb{C}$ for each $f(x_1,x_2) \in B_r(p)$

Like $\mathbb{C}$ is a topological space hence $\exists B_r(a,b) \in \tau$; an open subset of $\mathbb{C}$ for each $(a,b) \in \mathbb{C}$

Hence the inverse image of f

$$f^{-1}[f(x_1,x_2)] = f^{-1}[g(Prod_*(x_1,x_2)] \qquad ; (\text{where} \quad f^{-1} = Prod_*^{-1} g^{-1})$$

$$= f^{-1}[g(x_1 * x_2)]$$

$$= f^{-1}[g(x_1) \bullet g(x_2)]$$

$$= f^{-1}[Prod_{\bullet}(g(x_1), g(x_2))] = f^{-1}[Prod_{\bullet} g(x_1, x_2)] \in \mathbb{C} \qquad ; (\text{where} \quad f^{-1} = g^{-1} Prod_{\bullet}^{-1})$$

such that

$$f^{-1}(B_r(p)) = \{(x_1, x_2) \in \mathbb{C} \quad | \quad f(x_1, x_2) \in B_r(p); \quad \text{f is an homomorphism}\} \subseteq B_r(a, b) \subseteq \mathbb{C}$$

is an open subset of $\mathbb{C}$ because for a topological space $\exists B_r(a, b) \in \tau$; an open subset of $\mathbb{C}$ for each collection of $(x_1, x_2) \in \mathbb{C}$

1.4. The inverse function f^{-1} is continuous

Lets take $(x_1, x_2) \in B_r(a, b) = V_{\mathbb{C}} \subseteq \mathbb{C}$

Like

$$(x_1, x_2) = f^{-1}(f(x_1, x_2))$$

hence $\exists \quad f(x_1, x_2)$ for each (x_1, x_2) through inverse function f^{-1}

Like $\Re$ is a topological space hence exist an open set $B_r(p)$ for all

$$f(x_1, x_2) = g(x_1 * x_2) = g(x_1) \cdot f(x_2) \in \Re$$

therefore

$$f(V_{\mathbb{C}}) = \{g(x_1 * x_2) = g(x_1) \cdot g(x_2) \in \Re \quad | \quad f^{-1}(f(x_1, x_2)) \in V_{\mathbb{C}} \text{ such that f is an homomorphism}\}$$

is an open subset of $\Re$

Therefore f is an homeomorphism

2. Be Hausdorff.

Def. Points x and y in a topological space X can be separated by neighbourhoods if there exists a neighborhood U of x and a neighbourhood V of y such that U and V are disjoint ($U \cap V = \phi$). X is a Hausdorff space if any two distinct points in X are separated by neighbourhoods.

Proof.

Lets consider

$$B_{r1}(a,b) = \{(x_1,x_2) \in \mathbb{C} \quad | \quad d[(x_1,x_2),(a,b)] < r1\}$$

and

$$B_{r2}(c,d) = \{(x_1,x_2) \in \mathbb{C} \quad | \quad d[(x_1,x_2),(c,d)] < r2\}$$

Lets add both terms of distance

$$d[(x_1,x_2),(a,b)] = r1 \quad ;$$

$$d[(x_1,x_2),(c,d)] = r2$$

hence

$$d[(x_1,x_2),(a,b)] + d[(x_1,x_2),(c,d)] = r1 + r2$$

Also consider

$$d[(a,b),(c,d)] = r3$$

Lets consider that $a \neq c$ and $b \neq d$; such that $B_r(a,b) \cap B_r(c,d) = \phi$

Therefore

$$r1 + r2 < r3$$

Hence $\mathbb{C}$ is Hausdorff

3. Verify second countability axiom

Def. A second-countable space, also called a completely separable space, is a topological space whose topology has a countable base. More explicitly, a topological space T is second-countable if there exists some countable collection $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of open subsets of T such that any open subset of T can be written as a union of elements of some subfamily of $\mathcal{U}$.

Def. If $\mathcal{B}$ is a base for the topology τ of a space X, it satisfies the following properties:

(B1) The elements of $\mathcal{B}$ cover X, i.e., every point $x \in X$ belongs to some element of $\mathcal{B}$.

(B2) For every $B_1, B_2 \in \mathcal{B}$ and every point $x \in B_1 \cap B_2$, there exists some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

Proof.

By compact property about cover, $B1$ is satisfied with the elements $B_r(x_1, x_2) \subset c$

Lets take

$$(x_1, x_2) \in B_r(a, b) \cap B_r(c, d)$$

By subcover property of compact set $\mathbb{C}$; $\exists \in \tau \quad ; B_r(e, f) \subset B_r(a, b) \in \mathcal{B}$ such that

$$B_r(e, f) = \{(x_1, x_2) \in \mathbb{C} \quad | \quad d[(x_1, x_2), (e, f)] < r\} \quad ;$$

Hence

$$(x_1, x_2) \in B_r(e, f) \subseteq B_r(a, b) \cap B_r(c, d) = B_r(a, b)$$

by cover property with

$$(x_1, x_2) \in B_r(e, f) = B_r(a, b) \quad ;$$

$$(x_1, x_2) \in B_r(e, f) \subseteq B_r(a, b) \cap B_r(c, d) = B_r(a, b)$$

hence (B2) is satisfied

Therefore $\mathcal{B}$ is a base for τ

Like $B_r(x_1, x_2)_p$ has the index $p \in \{1, 2, 3, ...\}$; therefore $\mathcal{B}$ is countable

By subcover property $B_3 \subset B_1$; with B_1 on the cover

$$(x_1, x_2) \in B_1(a, b) \quad ;$$

Taking $B_3 \subseteq B_1$ and $B_3^* \subseteq B_1^*$ on the subcover such that B_3, $B_3^* \in c^*$

if $B_1 = B_3^*$ hence

$$B_3 \cup B_3^* = B_1 \in \tau$$

Therefore $\exists$ $B_r(x_1, x_2)_p$ such that

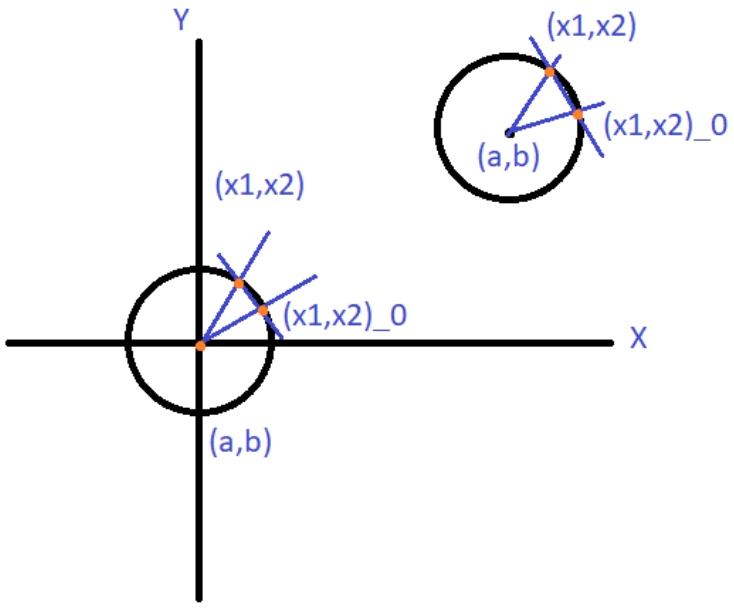
$$\mathcal{B} = \{B_r(x_1, x_2)_p\}_{p=1}^{\infty}$$

Hence $\mathbb{C}$ is second-countable

Therefore $\mathbb{C}$ is a topological manifold

4. f is holomorphic ref [1]

Lets use the scheme of open sets used to establish a meaning of derivative using secant on open sets and next definitions.



Derivative Scheme

$$f' = \begin{matrix} lim_{x_1 \to (x_1)_0} \\ lim_{x_2 \to (x_2)_0} \end{matrix} \left(\frac{f(x_1, x_2) - f(x_1, x_2)_0}{d[(x_1, x_2), (x_1, x_2)_0]} \right)$$

with

f an homomorphism

$$f(x_1, x_2) = g(Prod_*(x_1, x_2)) = g(x_1 * x_2) = g(x_1) \bullet g(x_2) = Prod_\bullet(g(x_1), g(x_2)) = Prod_\bullet g(x_1, x_2)$$

Lets obtain first the limit of $d[(x_1, x_2), (x_1, x_2)_0]$

$$\begin{matrix} lim_{x_1 \to (x_1)_0} \\ lim_{x_2 \to (x_2)_0} \end{matrix} d[(x_1, x_2), (x_1, x_2)_0] = \begin{matrix} lim_{x_1 \to (x_1)_0} \\ lim_{x_2 \to (x_2)_0} \end{matrix} \sqrt{(x_1 - (x_1)_0)^2 + (x_2 - (x_2)_0)^2}$$

$$= \sqrt{(lim_{x_1 \to (x_1)_0} x_1 - (x_1)_0)^2 + (lim_{x_2 \to (x_2)_0} x_2 - (x_2)_0)^2}$$

Hence for x_1

Def. The limit of f of x, as x approaches p, exists, and its equals $L = lim_{x \to p} f(x)$; if the following property holds:

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \Re)(0 < |x - p| < \delta \to |f(x) - L| < \epsilon)$$

Proof.

$$|x_1 - lim_{x_1 \to (x_1)_0} x_1| < \epsilon \quad ;$$

$$|x_1 - (x_1)_0| < \epsilon = \delta$$

Hence exist $(x_1)_0$ and L (and also for $(x_2)_0$)

Def. By noting that $|x - p|$ represents a distance; in the case of a function $f : S \times T \to \Re$ defined on $S \times T \subseteq \Re^2$: the limit of f as (x, y) approaches (p, q) is

$$L = lim_{(x,y) \to (p,q)} f(x, y)$$

if the following condition holds:

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in S)(\forall y \in T)(0 < \sqrt{(x - p)^2 + (y - q)^2} < \delta \to |f(x, y) - L| < \epsilon$$

Proof.

Hence

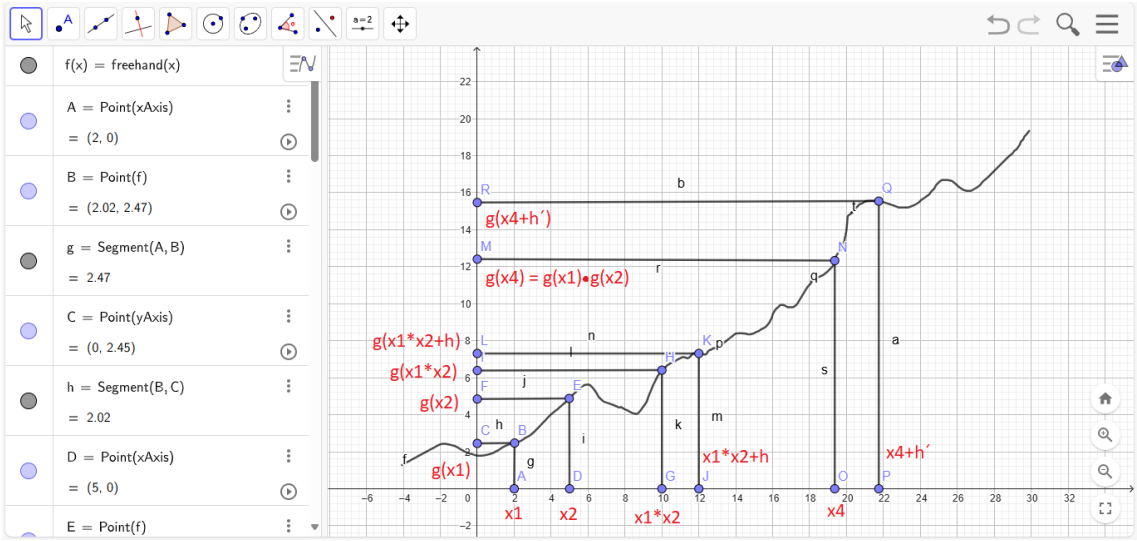
$$\left| \sqrt{[x_1 - (x_1)_0]^2 + [x_2 - (x_2)_0]^2} - \frac{lim_{x_1 \to (x_1)_0}}{lim_{x_2 \to (x_2)_0}} \sqrt{[x_1 - (x_1)_0]^2 + [x_2 - (x_2)_0]^2} \right| < \epsilon \quad ;$$

$$\left|\sqrt{[x_1-(x_1)_0]^2+[x_2-(x_2)_0]^2}\right|<\epsilon \quad ;$$

Hence

$$0<d[(x_1,x_2),(x_1,x_2)_0]<\epsilon=\delta$$

Now lets define derivative (see next scheme about derivative)



Scheme: Derivative

with

$$g(x_1)\bullet g(x_2)=g(x_4)$$

hence

$$x_4=g^{-1}(g(x_1)\bullet g(x_2))$$

Therefore derivative definition with g additive and proportional is

$$f'=\begin{matrix}lim_{x_1\to(x_1)_0}\\lim_{x_2\to(x_2)_0}\\lim_{d\to 0}\end{matrix}\left(\frac{f(x_1,x_2)-f(x_1,x_2)_0}{d[(x_1,x_2),(x_1,x_2)_0]}\right)=$$

$$\substack{lim_{x_1\to(x_1)_0}\\lim_{x_2\to(x_2)_0}\\lim_{d\to 0}}\Big(\frac{g(x_1 * x_2 + d) - g((x_1)_0 * (x_2)_0)}{d}\Big) =$$

$$\substack{lim_{x_1\to(x_1)_0}\\lim_{x_2\to(x_2)_0}\\lim_{d'\to 0}}\Big(\frac{g(g^{-1}(g(x_1) \bullet g(x_2)) + d') - g(g^{-1}(g((x_1)_0) \bullet g((x_2)_0)))}{d'}\Big)$$

such that

$$L = \Big(\frac{g(lim_{x_1\to(x_1)_0}x_1 * lim_{x_2\to(x_2)_0}x_2 + lim_{d\to 0}d) - g((x_1)_0 * (x_2)_0)}{lim_{d\to 0}d}\Big) =$$

$$\Big(\frac{g(g^{-1}(g(lim_{x_1\to(x_1)_0}x_1) \bullet g(lim_{x_2\to(x_2)_0}x_2)) + lim_{d'\to 0}d') - g(g^{-1}(g((x_1)_0) \bullet g((x_2)_0)))}{lim_{d'\to 0}d'}\Big)$$

Def. A function of a real variable $f(x)$ is differentiable at a point a of its domain, if its domain contains an open interval containing a, and the limit

$$L = lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

exist. This means that,

$$(\forall\epsilon > 0)(\exists\delta > 0)(\forall h \quad ; |h| < \delta \wedge h \neq 0 \to \exists f(a+h) \wedge$$

$$|L - \frac{f(a+h) - f(a)}{h}| < \epsilon$$

Proof.

If $L = 0$; taking as last definition hence

$$\Big|\frac{g(x_1 * x_2 + d) - g((x_1)_0 * (x_2)_0)}{d}\Big| =$$

$$\Big|\frac{g(g^{-1}(g(x_1) \bullet g(x_2)) + d') - g(g^{-1}(g((x_1)_0) \bullet g((x_2)_0)))}{d'}\Big| < \epsilon$$

hence

$$d = g(d') > |\frac{d}{(...)}| = |\frac{g(d')}{(...)}| > \frac{1}{\epsilon}$$

therefore

$$d = g(d') > \frac{1}{\epsilon}$$

therefore $\exists \delta > 0$ such that

$$\delta > d = g(d') > \frac{1}{\epsilon}$$

Hence $f(x_1, x_2) = g(x_1 * x_2) = g(x_1) \bullet g(x_2) \in \mathbb{C}$ is differentiable

Def. A function is holomorphic on an open set U if it is complex differentiable at every point of U.

Proof. Lets take

$$f(x_1, x_2) = g(Prod_*(x_1, x_2)) = Prod_\bullet(g(x_1), g(x_2))$$

such that

$\exists g(x_3 + d) = g(g^{-1}(g(x_1) * g(x_2)) + d') = g(x_4 + d')$; and $\exists |L - \frac{(...)}{d}| < \epsilon$;

with $L = lim_{x_1, x_2, d}[f(x_1, x_2)]$

such that

$$(x_1, x_2) \in B_r(a, b) = \{(x_1, x_2) \in \mathbb{C} | \quad d[(a, b), (x_1, x_2)] < r \quad \wedge \quad (x_1, x_2) \in dom(f)\} \in \tau$$

Hence $f(x_1, x_2)$ is differentiable at $B_r(a, b)$

Therefore $f(x_1, x_2)$ is holomorphic at $B_r(a, b)$

Def. A function f is holomorphic at a point z_0 if it is holomorphic on some neighborhood of z_0.

Proof.

Like $f(x_1, x_2)$ is holomorphic on $B_r(a, b)$ hence is holomorphic at $(x_1, x_2) \in B_r(a, b)$

Def. Orientability and orientations can also be expressed in terms of tangent bundle. The tangent bundle is a vector bundle, so it is a fiber bundle with structure group $GL(n, R)$. That is, the transition functions of the manifold induce transition functions on the tangent bundle which are fiberwise linear transformations. If the structure group can be reduced to the group $GL^+(n, R)$ of positive determinant matrices, or equivalently if there exists an atlas whose transition functions determine an orientation preserving linear transformation on each tangent space, then the manifold M is orientable. Conversely, M is orientable if and only if the structure group of the tangent bundle can be reduced in this way.

Proof.

Lets define first the complex vector bundle

Def. A complex vector bundle consist of:

1. topological spaces X (base space) and E (total space)

2. a continuous surjection $\pi : E \to X$ (bundle projection)

3. for every x in X, the structure of a finite-dimensional complex vector space on the fiber $\pi^{-1}(\{x\})$

where the following compatibility condition is satisfied: for every point p in X, there is an open neighborhood $U \subseteq X$ of p, a natural number k, and a homeomorphism

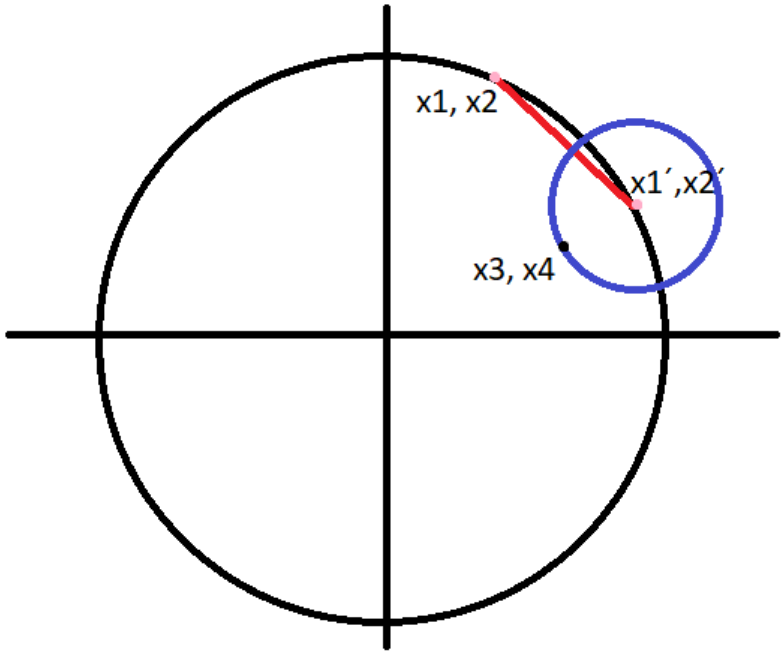
$$\varphi : U \times \Re^k \to \pi^{-1}(U)$$

such that for all x in U,

* $(\pi \circ \varphi)(x, v) = x$ for all vectors v in $\Re^k$, and

* the map $v \mapsto \varphi(x, v)$ is a linear isomorphism between the vector spaces $\Re^k$ and $\pi^{-1}(\{x\})$

Proof.



$B_r \times (B \times SB)$

Lets define first the sets E, B and F ref[1]

Lets take $E = \mathbb{C} :=$ Complex space a topological space like was proved before

Now lets define the circle on complex space

$$C = \{(x_1, x_2) \in \mathbb{C} \quad | \quad d[(a, b), (x_1, x_2)] = r\}$$

where (a, b) is the center of the circle

$S_pM = \{(x_1', x_2') \in C \quad | \quad d[(x_1, x_2), (x_1', x_2')] = 2r^2(1-cos\gamma) \neq 0; \quad \angle(x_1, x_2), (a, b), (x_1', x_2') \neq 0\}$

with $r = 10^i$

$i \in Z^+$ or $i \in Z^-$

$B = \{(x_1', x_2') \in \mathbb{C} \quad | \quad (x_1', x_2') \in \bigcup_{i \in Z^+ \cup Z^-} S_pM_i \quad \wedge \quad (x_1', x_2') \in \bigcap_{i \in Z^+ \cup Z^-} S_pM_i\}$

and

$F = B \times SB = \{((x_1', x_2'), (x_1, x_2)) \quad | \quad (x_1', x_2') \in B \wedge (x_1, x_2) \in SB\}$

where

$$SB = \{(x_1, x_2) \in C \quad | \quad (x_1, x_2) := \text{secant point to} (x_1', x_2') \in B\}$$

Property. Every fiber bundle $\pi : E \to B$ is an open map; since projections of products are open maps.

Now lets define π map like

$\pi = \{(x_1, x_2) \in \mathbb{C} \quad | \quad (x_1, x_2) \in C \wedge (x_1', x_2') \in S_pM \wedge (x_1', x_2') \in B \wedge (x_1 \neq x_1' \quad ; \quad x_2 \neq x_2')\}$

and

$\pi^{-1} = \{(x_1', x_2') \in B \quad | \quad x_1' = x_1 \quad ; \quad x_2' = x_2 \quad ; \quad (x_1, x_2) \in \mathbb{C}\}$

hence

$$\pi = e_{|_\pi} \wedge \pi^{-1} = e$$

i.e.

π is a restricted identity function

*) B is a topological space

First lets prove that B is a topological space

Now lets take $(x_1', x_2'), (x_3', x_4') \in B$ such that the open set

$$B_r(x'_1, x'_2) = \{(x'_3, x'_4) \in B \quad | \quad d[(x'_1, x'_2), (x'_3, x'_3)] < r\}$$

Hence if

$(x'_3, x'_4) \in B_r(x'_1, x'_2)$ and $(x'_3, x'_4) \in B'_r(x'_1, x'_2)$

therefore

$$(x'_3, x'_4) \in B_r(x'_1, x'_2) \cap B'_r(x'_1, x'_2) \in \tau$$

while if

$(x'_3, x'_4) \in B_r(x'_1, x'_2)$ or $(x'_3, x'_4) \in B'_r(x'_1, x'_2)$

$$(x'_3, x'_4) \in B_r(x'_1, x'_2) \cup B'_r(x'_1, x'_2) = B''_r(x'_1, x'_2) \in \tau$$

Hence

$$\forall B'' \subset \tau, \bigcup_{B_r \in B''} B_r \in \tau$$

therefore B is a topological space

Now lets prove that π is continuous

*) π is continuous

$\pi^{-1}(B_r(x'_1, x'_2)) = \{(x_1, x_2) \in E = \mathbb{C} \quad | \quad \pi(x_1, x_2) \in B_r(x'_1, x'_2)\}$

while

*) π is surjective

with $(x'_1, x'_2) \in B$ and $(x_1, x_2) \in E$ and

$d[(x_1, x_2), (x'_1, x'_2)] \neq 0 \quad \wedge \quad \angle(x_1, x_2), (a, b), (x'_1, x'_2) \neq 0$

therefore

$\forall (x_1', x_2') \in B; \exists (x_1, x_2) \in E$ such that

hence

$$\pi(x_1, x_2) = (x_1', x_2')$$

therefore π is surjective

*)

Besides taking into consideration that for every $(x_1', x_2') \in B$; the fiber π^{-1}; such that

$$\pi^{-1}(x_1', x_2') = (x_1, x_2) \in \mathbb{C}$$

has the structure of a complex space

*) Compatibility condition

Lets prove that φ is an homeomorphism function

$$\varphi : B_r(x_1', x_2') \times F \to \pi^{-1}(x_1', x_2')$$

$$\varphi := proj_2 \circ proj_2$$

Lets take $[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))]$ and $[(x_3', x_4'), ((x_3', x_4'), (x_3, x_4))]$

*) φ is one-to-one

$$\varphi[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] = proj_2 \circ proj_2[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))]$$

$$= proj_2((x_1', x_2'), (x_1, x_2)) = (x_1, x_2) \quad ;$$

$$\varphi[(x_3', x_4'), ((x_3', x_4'), (x_3, x_4))] = proj_2 \circ proj_2[(x_3', x_4'), ((x_3', x_4'), (x_3, x_4))]$$

$$= proj_2((x'_3, x'_4), (x_3, x_4)) = (x_3, x_4)$$

if $x_1 = x_3$ and $x_2 = x_4$

hence φ is one-to-one

*) φ is surjective

$$\varphi[(x'_1, x'_2), ((x'_1, x'_2), (x_1, x_2))] = proj_2 \circ proj_2[(x'_1, x'_2), ((x'_1, x'_2), (x_1, x_2))]$$

$$= proj_2((x'_1, x'_2), (x_1, x_2)) = (x_1, x_2)$$

hence

$$\exists[(x'_1, x'_2), ((x'_1, x'_2), (x_1, x_2))] \in B_r(x'_1, x'_2) \times F \quad ; \forall (x_1, x_2) \in \pi^{-1}(x'_1, x'_2)$$

hence φ is surjective

Hence φ is bijective

*) φ is continuous

Lets define first

$$\pi^{-1}(x'_1, x'_2) = \{(x_1, x_2) \in E \quad | \quad \pi(x_1, x_2) \in B\}$$

and lets define an open set on π^{-1}

$$B_r(x_1, x_2) = \{(x_3, x_4) \in E \quad | \quad d[(x_1, x_2), (x_3, x_4)] < r\} \subseteq \pi^{-1}(x'_1, x'_2)$$

*) π^{-1} is a topological space

Lets take

$(x_a, x_b) \in B_r(x_1, x_2)$ and $(x_a, x_b) \in B'_r(x_1, x_2)$

hence

$$(x_a, x_b) \in B_r(x_1, x_2) \cap B'_r(x_a, x_b) \in \tau$$

now lets take

$(x_a, x_b) \in B_r(x_1, x_2)$ or $(x_a, x_b) \in B'_r(x_1, x_2)$

hence

$$(x_a, x_b) \in B_r(x_1, x_2) \cup B'_r(x_a, x_b) = B_r(x_c, x_d) \in \tau$$

Therefore π^{-1} is a topological space

Now lets define the set

$$\varphi^{-1}(B_r(x_1, x_2))$$

$$= \{[(x'_1, x'_2), ((x'_1, x'_2), (x_1, x_2))] \in B_r(x'_1, x'_2) \times F \quad | \quad \varphi[(x'_1, x'_2), ((x'_1, x'_2), (x_1, x_2))] \in B_r(x_1, x_2)\}$$

Now lets prove that F is a topological space

with $F = B \times SB$

Early was established that B is a topological space

Now lets prove that SB is a topological space

Lets define

$$B_r(x_1, x_2) = \{(x_3, x_4) \in C \subseteq E \quad | \quad d[(x_1, x_2), (x_3, x_4)] < r\} \subseteq SB$$

Hence taking

$(x_a, x_b) \in B_r(x_1, x_2)$ and $(x_a, x_b) \in B'_r(x_1, x_2)$

Hence

$(x_a, x_b) \in B_r(x_1, x_2) \cap B'_r(x_1, x_2) \in \tau$

Now taking

$(x_a, x_b) \in B_r(x_1, x_2)$ or $(x_a, x_b) \in B'_r(x_1, x_2)$

Hence

$(x_a, x_b) \in B_r(x_1, x_2) \cup B'_r(x_1, x_2) = B''_r(x_1, x_2) \in \tau$

Therefore SB is a topological space

Now lets define an open set on F

$$B_r(x_1, x_2) = \{((x'_1, x'_2), (x_1, x_2)) \in B \times SB \quad | \quad d[(x'_1, x'_2), (x_1, x_2)] < r\}$$

*) $[(x'_1, x'_2), (x_1, x_2)] \in B_r(x_1, x_2)$ and $[(x'_1, x'_2), (x_1, x_2)] \in B'_r(x_1, x_2)$

therefore

$$[(x'_1, x'_2), (x_1, x_2)] \in B_r(x_1, x_2) \cap B'_r(x_1, x_2) \in \tau$$

*) $[(x'_1, x'_2), (x_1, x_2)] \in B_r(x_1, x_2)$ or $[(x'_1, x'_2), (x_1, x_2)] \in B'_r(x_1, x_2)$

therefore

$$[(x'_1, x'_2), (x_1, x_2)] \in B_r(x_1, x_2) \cup B'_r(x_1, x_2) = B''_r(x_1, x_2) \in \tau$$

Hence F is a topological space

Now lets define an open set on

$$B_r(x'_1, x'_2) =$$

$$= \{[(x'_3, x'_4), ((x'_3, x'_4), (x_3, x_4))] \quad | \quad (x_3, x_4) \in E \wedge d[(x'_1, x'_2), (x_3, x_4)] < r\}$$

$$\subseteq B_r(x_1', x_2') \times F$$

*)

$$[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] \in B_r(x_1', x_2')$$

and

$$[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] \in B_r'(x_1', x_2')$$

therefore

$$[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] \in B_r(x_1', x_2') \cap B_r'(x_1', x_2') \in \tau$$

*)

$$[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] \in B_r(x_1', x_2')$$

or

$$[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] \in B_r'(x_1', x_2')$$

therefore

$$[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] \in B_r(x_1', x_2') \cup B_r'(x_1', x_2') = B_r''(x_1', x_2') \in \tau$$

Hence $B_r(x_1', x_2') \times F$ is a topological space

Like

$$(x_3, x_4) \in B_r(x_1, x_2) \in \tau_{\pi^{-1}}$$

and

φ is surjective hence $\exists$

$$[(x_1', x_2'), ((x_1', x_2'), (x_1.x_2))] \in B_r(x_1', x_2')$$

$\forall (x_3, x_4) \in B_r(x_1, x_2) \in \tau_{\pi^{-1}}$

therefore

$$\varphi^{-1}(B_r(x_1, x_2)) =$$

$$= \{[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] \in B_r(x_1', x_2') \subseteq B_r(x_1', x_2') \times F \quad |$$

$$\varphi[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] = (x_1, x_2)\}$$

therefore

$$\varphi^{-1}(B_r(x_1, x_2)) \in \tau_{B_r(x_1', x_2') \times F}$$

is an open set of

$$B_r(x_1', x_2') \times F$$

therefore φ is continuous

*) φ^{-1} is continuous

Lets define the set

$$\varphi(B_r(x_1', x_2')) = \{(x_1, x_2) \in \pi^{-1}(x_1', x_2') \quad | \quad \varphi^{-1}(x_1, x_2) \in B_r(x_1', x_2') \subseteq B_r(x_1', x_2') \times F\}$$

Lets take

$$(x_1, x_2) \in B_r(x_1, x_2)$$

Like

$$(x_1, x_2) = proj_2 \circ proj_2[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] = \varphi[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] \in B_r(x_1, x_2)$$

Now lets develop the composition function

$$proj_2[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] = proj_2^{-1}(x_1, x_2) \quad ;$$

$$[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] = proj_2^{-1} \circ proj_2^{-1}(x_1, x_2) = \varphi^{-1}(x_1, x_2) \in B_r(x_1', x_2') \quad ;$$

Hence

$$\varphi(B_r(x_1', x_2')) = \{(x_1, x_2) \in B_r(x_1, x_2) \subseteq \pi^{-1}(x_1', x_2') \quad | \quad \varphi^{-1}(x_1, x_2) \in B_r(x_1', x_2') \subseteq B_r(x_1', x_2') \times F\}$$

Hence

φ^{-1} is continuous

hence

φ is an homeomorphism

*) $(\pi \circ \varphi)(x, v) = x \quad ; \quad v \in \Re^k$

$$(\pi \circ \varphi)[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))] = \pi \circ proj_2 \circ proj_2[(x_1', x_2'), ((x_1', x_2'), (x_1, x_2))]$$

$$= \pi \circ proj_2((x_1', x_2'), (x_1, x_2))$$

$$= \pi(x_1, x_2)$$

$$= \pi(\pi^{-1}(x_1', x_2'))$$

$$= (x_1', x_2')$$

*) the map $v \mapsto \varphi(x, v)$ is a linear isomorphism

Def. Let V and W be vector spaces over the same field K. A function $f : V \to W$ is said to be a linear map if for any two vectors $u, v \in V$ and an scalar $c \in K$ the following two conditions are satisfied:

* Additivity / operation of addition

$f(u + v) = f(u) + f(v)$

* Homogeneity of degree 1 / operation of scalar multiplication

$f(cu) = cf(u)$

Def. A complex structure on a real vector space V is a real linear transformation

$$J : V \to V$$

such that

$$J^2 = -id_V$$

A complex structure allows one to endow V with the structure of a complex vector space. Complex scalar multiplication can be defined by

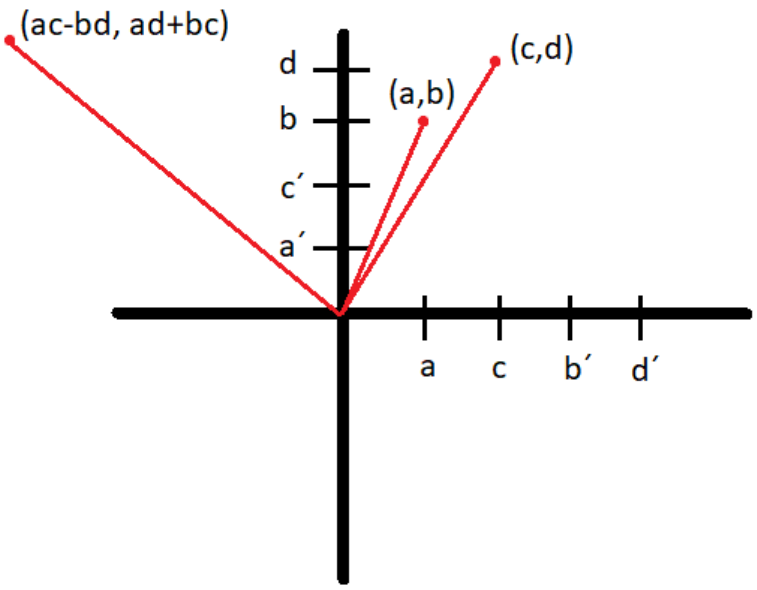
$$(x + iy)v = xv + yJ(v)$$

for all real numbers x, y and all vectors $v \in V$. The structure of a complex vector space which we denote V_J.

One can define J on pairs e, f of basis vectors by $Je = f$ and $Jf = -e$ and then extend by linearity to all of V. If $v_1, ..., v_n)$ is a basis for the complex vector space V_J then $(v_1, Jv_1, ..., v_n, Jv_n)$ is a basis for the underlying real space.

Proof.

Lets remember Argand´s product



Argand´s product

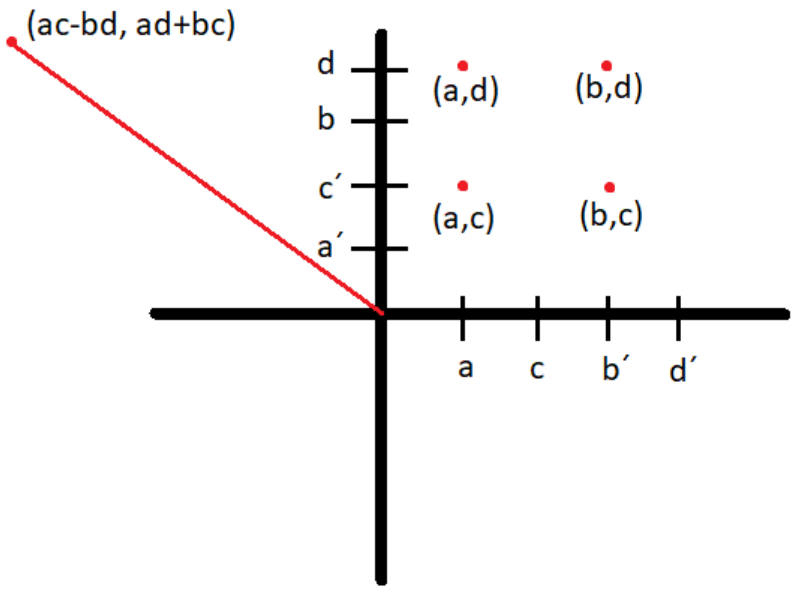
With

$E = \mathbb{C}$

$$\bullet_{\mathbb{C}} : E \times E \to E \quad ;$$

$$(a, b) \bullet_{\mathbb{C}} (c, d) = (ac - bd, ad + bc) \in \mathbb{C}$$

and lets define the result with Add_{map} and $Prod_{map}$ functions

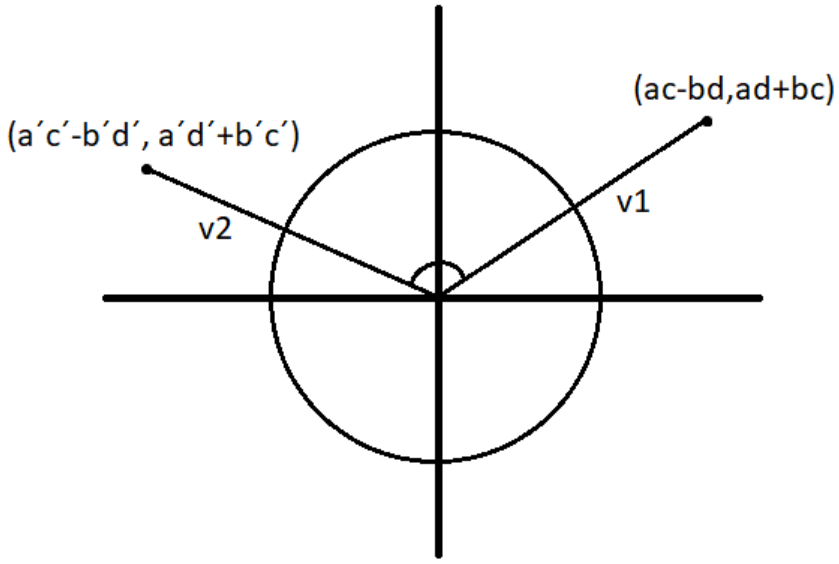


Argand´s product and pairs of $prod_{map}$ and add_{map}

with $(a, c), (b.d), (a, d)$ and $(b, c) \in \mathbb{C}$;

$$(ac - bd, ad + bc) = [Prod(a, c) - Prod(b, d), Prod(a, d) + Prod(b, c)]$$

$$= [Add_{map}(Prod(a, c), -Prod(a, c)), Add_{map}(Prod(a, d), Prod(b, c))]$$



Dot product

Now taking into account that

$$v_1 = (ac - bd, ad + bc) = (a, b) \bullet_{\mathbb{C}} (c, d) \quad ;$$

$$v_2 = (a'c' - b'd', a'd' + b'c') = (a', b') \bullet_{\mathbb{C}} (c', d')$$

considers the definition of dot product

$$v_1 \bullet_{\Re} v_2 = ||v_1|| \cdot ||v_2|| cos(\theta)$$

with

$$||v_1|| = \sqrt{(ac - bd)^2 + (ad + bc)^2} \quad ;$$

$$||v_2|| = \sqrt{(ac - bd)^2 + (ad + bc)^2}$$

and

$$\theta = \angle v_1 O v_2$$

Else consider the definition of homomorphism

$$f(A * B) = f(A) \cdot f(B) \quad ;$$

Hence with

$f := Add_{map}$;

$* := \bullet_{\mathbb{C}}$;

and

$\cdot :=$ arithmetic product

$$Add_{map}((a, b) \bullet_{\mathbb{C}} (c, d)) = Add_{map}(a, b) \cdot Add_{map}(c, d) \quad ;$$

$$Add_{map}(ac - bd, ad + bc) = Add_{map}(a, b) \cdot Add_{map}(c, d) \quad ;$$

$$ac - bd + ad + bc = (a + b) \cdot (c + d) \quad ;$$

$$ac - bd + ad + bc = ac + bd + ad + bc$$

Hence homomorphism is equal through $=_A$ (see ref[2, 3, 4, 5])

$$-bd =_A bd$$

i.e.

$$-bd < bd$$

While the elements

$$[(x_1', x_2'), (x_1, x_2)] \in B \times SB = F$$

that establish a secant line on the circle C when

$$x_1 = x_a' \quad ;$$

$$x_2 = x_b'$$

with

$$[(x_a', x_b'), (x_a, x_b] \in B \times SB = F$$

define the addition like

$$[(x_1', x_2'), (x_1, x_2)] + [(x_a', x_b'), (x_a, x_b)] = [(x_1', x_2'), (x_a, x_b)]$$

Hence

$$proj_2 : \mathbb{C} \times \mathbb{C} \to \pi^{-1}(\{x\})$$

establish

$$proj_2[c[(x_1', x_2'), (x_1, x_2)] + [(x_a', x_b'), (x_a, x_b)]] = proj_2[[c(x_1', x_2'), c(x_1, x_2)] + (x_a', x_b'), (x_a, x_b)]$$

$$= proj_2[[(cx_1', cx_2'), (cx_1, cx_2)] + [(x_a', x_b'), (x_a, x_b)]]$$

$$= proj_2[(cx_1', cx_2'), (cx_1, cx_2)] + proj_2[(x_a', x_b'), (x_a, x_b)]$$

$$= proj_2[c(x'_1, x'_2), c(x_1, x_2)] + proj_2[(x'_a, x'_b), (x_a, x_b)]$$

$$= c \cdot proj_2[(x'_1, x'_2), (x_1, x_2)] + proj_2[(x'_a, x'_b), (x_a, x_b)]$$

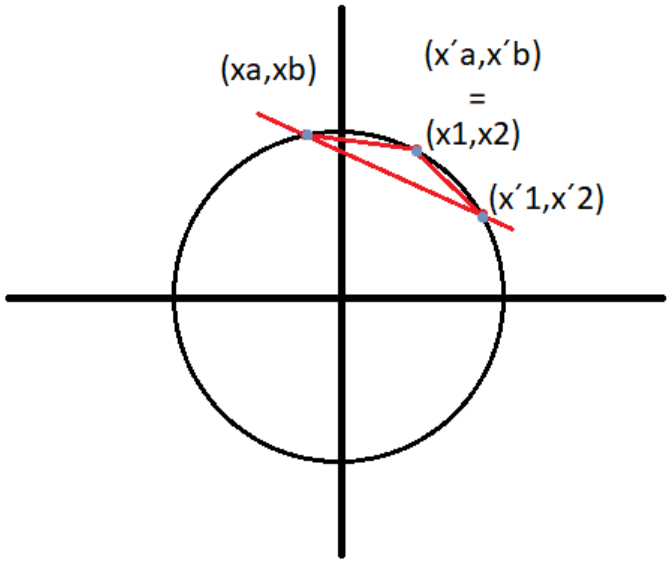
$$= c(x_1, x_2) + (x_a, x_b)$$

$$= c(x'_a, x'_b) + (x_a, x_b)$$

$$= (cx'_a, cx'_b) + (x_a, x_b)$$

establishing the line

$$[(cx'_a, cx'_b), (x_a, x_b)] \in B \times SB = F$$



linear map

Now lets prove that $proj_2$ is an isomorphism

Def. A partial isomorphism between two sets P and Q is bijective function $h : X \to Q$ with $X \subseteq P$.

Proof.

*) $proj_2$ is one - to - one

Taking

$$proj_2[(x'_1, x'_2), (x_1, x_2)] = (x_1, x_2) \quad ;$$

$$proj_2[(x'_a, x'_b), (x_a, x_b)] = (x_a, x_b)$$

hence if

$x_1 = x_a$ and $x_2 = x_b$

$proj_2$ is one- to-one

*) $proj_2$ is surjective

Consider

$$proj_2[(x_1', x_2'), (x_1, x_2)] = (x_1, x_2)$$

hence

$\exists[(x_1', x_2'), (x_1, x_2)] \in B \times SB = F \quad ; \forall (x_1, x_2) \in \pi^{-1}(\{x\})$

therefore $proj_2$ is surjective

therefore $proj_2$ is bijective

hence the map $proj_2$ is a linear isomorphism on a complex space $\mathbb{C} \times \mathbb{C}$ to $\pi^{-1}(\{x\})$

Taking into consideration early arguments hence

The complex scalar multiplication with $v = (a, b) \in \mathbb{C}$ and $x, y \in \Re$ is defined by

$$(x + iy)v = xv + yJ(v)$$

with $J := e_{\Re}$ hence

$$J : V \to V$$

such that

$$e_{\Re}^2 = [Add_{map}(1, 0)]^2 > -Add_{map}(1, 0)$$

hence

$$(1+0)^2 > -(1+0)$$

i.e. (see ref[2,3,4])

$$1 =_A -1$$

and

$(1,0) = e_{\mathbb{C}} \in \mathbb{C}$

hence

$$(x+iy)(a,b) = x(a,b) + ye_{\Re}(a,b) \quad ;$$

$$x(a,b) + iy(a,b) = x(a,b) + y(a,b) \quad ;$$

simplifying

$$iy(a,b) = y(a,b) \quad ;$$

and again

$$i(a,b) = (a,b) \quad ;$$

$$(\sqrt{-1})(a,b) = (a,b) \quad ;$$

hence the square is

$$(-1)(a,b)^2 = (a,b)^2$$

Like before was established that

$$(a,b) \bullet_{\mathbb{C}} (c,d) = (ac - bd, ad + bc)$$

hence

$$(a,b) \bullet_{\mathbb{C}} (a,b) = (a^2 - b^2, 2ab)$$

Like = (equal) sign have only the meaning given in arithmetic and geometry established before [see ref[2,3,4]]

$$(a,b)^2 \neq (a,b) \bullet_{\mathbb{C}} (a,b)$$

instead

$$(a,b)^2 = (a,b) \cdot (a,b) = ||(a,b)||^2 cos(\theta)$$

with $\theta = 0$ hence

$$(a,b)^2 = (a,b) \cdot (a,b) = ||(a,b)||^2 = a^2 + b^2$$

and the relation of$(a,b)^2$ with Argand´s product was considered like an "etiquette" error on the way during the development of the first texts of the complex field.

Hence

$$-(a,b)^2 = -(a^2 + b^2) = (a^2 + b^2)$$

have the meaning given by $=_A$ therefore

$$-(a^2 + b^2) =_A (a^2 + b^2)$$

i.e.

$$-(a^2 + b^2) < (a^2 + b^2)$$

Hence the unique real vector space V where J is defined for the linear complex structure is $\Re^2$

Def. In an algebraic structure an *automorphism* is simply a bijective homomorphism of an object into itself.

Proof.

Lets see that

$$e_{\Re} : \mathbb{C} \to \mathbb{C} \quad ;$$

establish the homomorphism

$$e_{\Re}[(a,b) * (c,d)] = e_{\Re}(a,b) \cdot e_{\Re}(c,d)$$

therefore

$$[(a,b) * (c,d)] = (a,b) \cdot (c,d)$$

if

$*$ is the same product as $\cdot$ hence

$e_{\Re}$ is an homomorphism

Like $e_{\Re}$ is bijective hence $V = \Re^2$ endowed of a complex algebra by ref[2, 3, 4] is an automorphism (i.e. $V = \mathbb{C}$)

Hence $e_{\Re}$ on $\mathbb{C}$ is a linear complex structure

Now lets define the relation between

$[(x_1', x_2'), (x_1, x_2)]$ and $(a,b) \bullet_{\mathbb{C}} (c,d) = (ac - bd, ad + bc)$

Remember that

(x_1', x_2') and $(x_1, x_2) \in C :=$ a circle

now remember the equation of a circle

$$x_1^2 + x_2^2 = r^2$$

and take into consideration that

$\sqrt{(ac-bd)^2+(ad+bc)^2} = d$

with

$d = r$

hence

$x_1 = ac - bd$

and

$x_2 = ad + bc$

while the angle between both points can be calculated by

$$(x_1', x_2') \cdot (x_1, x_2) = ||(x_1', x_2'|| \cdot ||(x_1, x_2)||cos(\theta)$$

or

Chord length $= d[(x_1', x_2'), (x_1, x_2)] = 2rsin(\frac{\theta}{2})$

and clearing θ.

Def. The general linear group over the field of complex numbers, $GL(n, \mathbb{C})$, is a complex Lie group. The set of all real matrices forms a real Lie subgroup. These correspond to the inclusions

$$GL(n, \Re) < GL(n, \mathbb{C}) < GL(2n, \Re)$$

with

$$GL(n, \mathbb{C}) = \{A \in GL(2n, \Re) \quad | \quad AJ = JA\}$$

Complex n-dimensional matrices can be characterized as real 2n - dimensional matrices that preserve a linear complex structure, that is, matrices that commute with a matrix J such that $J^2 = -I$

The Lie algebra corresponding to $GL(n,\mathbb{C})$ consists of all $n \times n$ complex matrices with the commutator serving as the Lie bracket $[J, A] = 0$.

Def. Topological definition. A Lie group can be defined as a (Hausdorff) topological group that, near the identity element, looks like a transformation group, with no reference to differentiable manifolds. First, we define an immersely linear Lie group to be a subgroup G of the general linear group $GL(n,\mathbb{C})$ such that

1. For some neighborhood V of the identity element e in G, the topology on G is the subspace topology of $GL(n,\mathbb{C})$ and V is closed in $GL(n,\mathbb{C})$.

2. G has at most countably many connected components.

Def. A topological group G, is a topological space that is also a group such that the group operation (in this case product):

$$\cdot : G \times G \to G \quad ; \quad (x, y) \mapsto xy$$

and the inversion map:

$$^{-1} : G \to G \quad ; \quad x \mapsto x^{-1}$$

are continuous. Here $G \times G$ is viewed as a topological space with the product topology. Such topology is said to be compatible with the group operations and is called a group topology.

Def. (see Ref[1]) A real Lie group is a group that is also a finite-dimensional real smooth manifold, in which the group operations of multiplication and inversion are smooth maps. Smoothness of the group multiplication

$$\mu : G \times G \to G \quad ; \quad \mu(x, y) = xy$$

means that μ is a smooth mapping of the product manifold $G \times G$ into G. The two requirements can be combined into the single requirement that the mapping

$$(x, y) \mapsto x^{-1}y \quad ;$$

be a smooth mapping of the product manifold into G.

Proof.

Lets take the term of charge established on ref[1] to establish the meaning of Lie group

With

$$q(t) = \frac{n}{q^{e^-}} \cdot \frac{h\nu}{m_e c^2} \quad ; \quad \text{where} \quad c = \frac{dx}{dt}$$

lets define next equation

$$q(t) = A\frac{1}{(\frac{dx}{dt})^2} = A(\frac{dt}{dx})^2 = A(\frac{dt}{dx_1})(\frac{dt}{dx_2})$$

with $A :=$ constant scalar; where the term of the right is represented in terms of x_1 and x_2 variables of $(x_1, x_2) \in \mathbb{C}$

With

$$\frac{dx_1}{dt} \quad \wedge \quad \frac{dx_2}{dt} \in \Re$$

Lets take the product of both terms

$$Prod(\frac{dx_1}{dt}; \frac{dx_2}{dt}) = (\frac{dx_1}{dt}) \cdot (\frac{dx_2}{dt})$$

with

$$f(x_1, x_2) = g \cdot Prod(x_1, x_2) = Prod(g(x_1), g(x_2))$$

a homomorphism hence

$$f(x_1, x_2) = e_{\Re} \cdot Prod(x_1, x_2) = Prod(e_{\Re}(x_1), e_{\Re}(x_2))$$

where $\frac{dx}{dt}$ represent the quotient of x variable by t

Hence

$$q(t) = A\frac{1}{f(\frac{dx_1}{dt}, \frac{dx_2}{dt})}$$

Now lets multiply by -1

$$q(t)^{-1} = \frac{1}{q(t)} = \frac{1}{A} \cdot \frac{dx_1}{dt} \cdot \frac{dx_2}{dt}$$

Like $q(t)^{-1} \in Q$ and the product between scalars $(\cdot)$ is commutative hence with $dt = T$; $dx_1 = X_1$ and $dx_2 = X_2$

$$q'(t) = AT^2 \cdot q(t)^{-1} = X_1 \cdot X_2 \in Q(t)'$$

also forms a group.

Like on ref[yang-mills] was proved that $q(t) \in Q(t)$ forms a topological and a differential manifold; besides also a real Lie group with $Prod(q_1, q_2)$ the continuous group operation; hence $Q(t)'$ is also a Lie group (a differential manifold and a topological manifold).

Hence only the properties about the $GL(n, \mathbb{C})$ will be established next.

Like

$$Prod(q_1', q_2') \quad : \quad \mathbb{C} \quad \rightarrow \quad \Re \quad ;$$

$$[(X_1 \cdot X_2), (Y_1 \cdot Y_2)] \mapsto (X_1 \cdot X_2) \cdot (Y_1 \cdot Y_2)$$

with $q'(t) \in \Re$; define the matrix multiplication hence the matrix degree of the GL set is $n = 1$; with $q'(t)$ the matrix of 1×1.

*) the matrix $q'(t)$ is invertible

Def. An $n \times n$ matrix A is called invertible if there exists an $n \times n$ square matrix B such that

$$AB = BA = I_n$$

where I_n denotes the $n \times n$ identity matrix and the multiplication used is ordinary matrix multiplication.

Proof.

Lets take $q'(t) \in Q'(t)$

$$q'(t) \cdot q' \quad ^{-1}(t) = q' \quad ^{-1} \cdot q'(t) = e_{\Re} \in Re$$

Hence $q'(t)$ is invertible

Hence $Q(t)$ is the automorphism group.

*) $JA = AJ$

Like $J = e_{\Re}$

hence

$$e_{\Re} \cdot q'(t) = q'(t) = q'(t) \cdot e_{\Re} \in Q'(t) \subseteq \Re$$

*) Inclusion map

Def. In mathematics if A is a subset of B, then the inclusion map is the function $\imath$ that sends each element x of A to x, treated as an element of B:

$$\imath : A \to B \quad ; \quad \imath(x) = x$$

Proof.

Lets analyze the inclusion of GL groups

with $q' \in Q' \subseteq \Re$ and a matrix 1×1 hence

$$q \in GL(1, \Re)$$

Now remember that

$\mathbb{C} = \Re \times \Re$ endowed of $-a =_A a$ properties

hence with cartesian product

$$A \times B = \{ \quad (a,b) \quad | \quad a \in A \wedge b \in B \quad \}$$

hence

$$q' \in GL(1,\Re) \subseteq \Re \wedge q^* \in GL(1,\Re) \subseteq \Re$$

therefore providing each $\Re$ with $-a =_A a$

$$(q', q^*) \in \mathbb{C}$$

hence

$$(q', q^*) \in GL(1,\mathbb{C})$$

Later with

$$p, p' \in GL(1,\Re)$$

hence

$$[(q', q^*), (p', p^*)] \in \mathbb{C} \times \mathbb{C} = \mathbb{C}^{\not\perp}$$

hence with A defined like

$$A = \begin{vmatrix} q' & q^* \\ p' & p^* \end{vmatrix} \quad ;$$

$$A \in GL(2,\mathbb{C})$$

therefore the GL are link by

$$GL(1,\Re) \quad \underrightarrow{GL(1,\Re) \times GL(1,\Re)} \quad GL(1,\mathbb{C}) \quad \underrightarrow{GL(1,\mathbb{C}) \times GL(1,\mathbb{C})} \quad GL(2,\mathbb{C}^{\not\models})$$

by the other hand lets consider

$\Re :=$ set of real numbers

and

$\Re_A :=$set of abstract equality line segments on $\Re$

where

$$\Re_A = \{ \quad a \in \Re \quad | \quad -a =_A a \quad ; \quad (-a < a) \quad \}$$

define a non reflexive and non symmetric set

Like every number must be related only with itself hence the cardinality of $\Re$ through $=_A$ is $\frac{1}{2}$

Here inclusion function establish

$$\Re_A \subseteq \Re$$

hence with

$$x \in \Re_A \quad ;$$

$$i(x) = x \in \Re \quad ;$$

*) Orientation

Later the orientation of the set is define by the inequality $<$ through

$a \in [A, B]$ where

if

$A < B$; hence a has orientation to the left

and if

$A > B$; hence a has orientation to the right

On $\Re^2$ the orientation could be defined through vector projection of line segment $[(x_1', x_2'), (x_1, x_2)] \in F$;

With

Chord length $= d[(x_1', x_2'), (x_1, x_2)] = 2rsin(\frac{\theta}{2})$

hence clearing

$$\theta = 2arcsin[\frac{d[(x_1', x_2'), (x_1, x_2)]}{2r}] \quad ;$$

with

$$r = \sqrt{x_1^2 + x_2^2} = \sqrt{x_1'^2 + x_2'^2} \quad ;$$

and

$$x_1' - x_1 = y_1 \quad ;$$

$$x_2 - x_2' = y_2$$

and using the cosine theorem hence

$$y_2^2 = y_1^2 + d[(x_1', x_2'), (x_1, x_2)]^2 - 2y_1 \cdot d[(x_1', x_2'), (x_1, x_2)]cos(\gamma) \quad ;$$

and clearing

$$\phi = arccos[\frac{y_2^2 - y_1^2 - d[(x_1', x_2'), (x_1, x_2)]^2}{2y_1 \cdot d[(x_1', x_2'), (x_1, x_2)]}]$$

with $\theta = [0, 90]$;

Hence orientation could be define with $(x_1, x_2) \in SB$ and $(x'_1, x'_2) \in B$

when x'_1, $x_1 < 0$ hence

$y_1 < 0$ and angle must be considered through $-y_1 =_A y_1$

when $y_2 < 0$ the angle must be calculated with $-y_2 =_a y_2$

hence a, b and c always can be considered like positive numbers.

*) Transition function. **Prop.** The transition functions of the manifold induce transition functions on the tangent bundle which are fiberwise linear transformations. [See ref. 1]

Def. Suppose that $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two charts for a (topological) manifold M such that $U_\alpha \cap U_\beta$ is non-empty. The transition map

$\tau_{\alpha,\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$

is the map defined by

$\tau_{\alpha,\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$

Note that φ_α and φ_β are both homeomorphisms, hence the transition map $\tau_{\alpha,\beta}$ is also a homeomorphism.

Def. A chart for a topological space M is a homeomorphism φ from an open subset U of M to an open subset of a Euclidean space. The chart is traditionally recorded as the ordered pair (U, φ).

Def. An atlas for a topological space M is an indexed family $\{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ of charts on M which covers M (that is, $\bigcup_{\alpha \in I} U_\alpha = M$.

Proof.

Lets take two open sets $B_r(x_a, x_b)$ and $B_r(x_c, x_d)$ (for $x_a = x_c$ and $x_b = x_d$ with different radius or $x_a \neq x_c$ and $x_b \neq x_d$) on the topological space $\mathbb{C}$.

Now consider next definition about φ functions to define the chart $(\varphi_\alpha, U_\alpha)$.

Def. Let be $\varphi_i \in V_\varphi$ an homeomorphism with canonical base $\mathcal{B} = \{e_1, e_2, ..., e_n\} \in V$ with $e_i = (0, 0, ..., 0, 1, 0, ..., 0)$ meaning that the 1 is in the i-th position and all other entries are zero; where V is a n-dimensional vector space over a field F and

$V = F^n$ be the set of all n-tuples $(a_1, a_2, ..., a_n)$; such that the V is linear independent (no e_i can be written as a linear combination of the others); is a spanned space (any vector $v = (a_1, ..., a_n)$ can be expressed as: $v = a_1e_1 + a_2e_2 + ... + a_ne_n$ and is unique (the default basis represent a vector as a coordinate list).

Also consider next definition

Def. In n-dimensions rotations happen in planes spanned by pairs of coordinate axes. A rotation in the plane defined by coordinates i and j (with rotation angle θ) has the form

$$R_{(i,j)}(\theta) = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \cos\theta & \cdots & -\sin\theta & & \\ & & \vdots & \ddots & \vdots & & \\ & & \sin\theta & \cdots & \cos\theta & & \\ & & & & & \ddots & \end{bmatrix}$$

Any rotation in n-dimensions can be written as a product of such givens rotation matrices as:

$$R = \Pi_{(i,j)} R(i,j)(\theta_{ij})$$

where each θ_{ij} is the rotation angle in the plane spanned by the axes i and j.

Taking v consider that

$$R_{(ij)}(\theta)v = v$$

Hence orientation has the definition given early.

With $(x_a, x_b) \in SB$ and $(x'_a, x'_b) \in B$; $L = [(x'_1, x'_2), (x_1, x_2)]$ is a line segment and definition of $proj_2$ function preserves the linear transformation on the manifold $\mathcal{C}$; besides dot product definition also is preserved; i.e.

$$L \in F = B \times SB \quad ;$$

With

$(\varphi_\alpha, U_\alpha)$ and (φ_β, U_β) hence with $U_\alpha \subseteq U_\beta$

therefore

$$\tau_{\alpha,\beta}(U_\alpha \cap U_\beta) = \varphi_\beta \circ \varphi_\alpha \circ \varphi_\alpha^{-1}(U_\alpha \cap U_\beta)$$

$$= \varphi_\beta(U_\alpha \cap U_\beta) \subseteq U_\beta$$

Hence φ is an homeomorphism.

Hence exist an atlas of the manifold $\mathbb{C}$ whose transition functions; preserving orientation and linear transformation; induce transition functions $\tau_{\alpha\beta}$ on the secant bundle F.

*) **Def.** A fiberwise linear transformation is a map

$$T : E \to E'$$

between two vector bundles $\pi : E \to M$ and $\pi' : E' \to M$ over the same base space M such that:

1. Base point preserving: $\pi'(T(v)) = \pi(v) \quad ; \quad \forall v \in E$

2. Linearity on each fiber; for each $x \in M$, the restriction $T_x := T_{|_{E_x}} : E_x \to E'_x$ is a linear transformation of vector spaces.

Proof.

Like

$$B_r(x_a, x_b) \subseteq E$$

and

$$B_r(x_c, x_d) \subseteq E'$$

with

$$\tau_{\alpha\beta} : E \to E'$$

is a map between two vector bundles $\pi : E \to B$ and $\pi' : E \to B$ over the same base space B such that:

$$\tau_{\alpha\beta}(x_a, x_b) = (x_a, x_b) \quad ; \quad \forall (x_a, x_b) \in E$$

hence

$$\pi'(\tau_{\alpha\beta}(x_a, x_b)) = \pi'(x_a, x_b) = \pi(x_a, x_b) \quad ;$$

with the fiber

$$\pi^{-1}(x_a, x_b) = (x'_a, x'_b) \in B \quad ;$$

the restriction

$$\tau_{(x_a, x_b)} := \tau_{|_{(x'_a, x'_b)}} : E_{(x_a, x_b)} \to E'_{(x_a, x_b)}$$

establish the transition map used to define the cartesian product $B \times SB = F$.

Hence $\tau_{\alpha\beta}$ preserves the linear transformation on F.

Hence $\tau_{\alpha\beta}$ is a fiberwise linear transformation.

Def. If a map is real-analytic then is smooth.

Proof.

Def. A *real analytic* function is an infinitely differentiable function such that the Taylor series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

at any point x_0 in its domain converges to $f(x)$ for x in a neighborhood of x_0.

Proof.

Lets define the differentiability of $f(x_1, x_2)$ considering next properties

*)

$$f(x_1, x_2) = g(Prod_*(x_1, x_2)) = Prod_*(g(x_1, x_2))$$

i.e. $f(x_1, x_2)$ is an homomorphism

*)

$$g : \Re \to \Re$$

i.e. g is a function of one variable

*)

$$Prod(x_1, x_2) : \Re^2 \to \Re$$

i.e. $Prod$ is a function of two variables

*)

$$x_2 \neq f(x_1)$$

i.e. x_2 is independent of x_1

With $f(x, y) = z$, $x_1 = x$ and $x_2 = y$ lets see that like $f(x, y)$ is a composition function of g and $Prod$ and taking the homomorphism property, can be established that the derivatives of $f(x, y)$ can be calculated like derivatives of one variable.

Now consider the expression of Taylor series and lets substitute the term using the properties established early.

$$T(x, y) = \sum_{n=0}^{\infty} (\frac{f^{(n)}(x, y)}{n!})[Prod(x, y) - Prod(x_0, y_0)]^n$$

$$= \sum_{n=0}^{\infty} (\frac{(g(Prod(x, y)))^{(n)}}{n!})[Prod(x, y) - Prod(x_0, y_0)]^n$$

$$= \sum_{n=0}^{\infty} (\frac{(Prod(g(x,y)))^{(n)}}{n!})[Prod(x,y) - Prod(x_0,y_0)]^n$$

if g is smooth (infinitely differentiable) hence $f(x,y)$ is smooth

hence $f(x,y)$ is real analytic

Now lets take the φ homeomorphism used to define the transition map

With $\varphi \in \mathbb{C}$

Hence

$$\varphi = g(Prod(x,y)) = Prod(g(x,y))$$

Hence

$$T(x,y) = \sum_{n=0}^{\infty} (\frac{\varphi^{(n)}(x,y)}{n!})[Prod(x,y) - Prod(x_0,y_0)]^n$$

Hence the map is real analytic

Hence the map is smooth

Def. If a smooth map is C^k (infinitely differentiable) for any k then its analytic atlas is an smooth atlas.

Proof.

Like φ is a smooth (infinitely differentiable) hence the map $\tau_{\alpha\beta}$ establish an analytic atlas

Hence the atlas is smooth

Def. A given topological space is a smooth manifold if a smooth atlas is specified.

Proof.

Like $\mathbb{C}$ is a topological space

and

$\tau_{\alpha\beta}$ is a smooth atlas

Hence

$\mathbb{C}$ is a smooth manifold

Def. If a map between open subsets of R^{2n} is real-analytic then it is an holomorphic map between open subsets of C^n.

Note.

If a map $\tau_{\alpha\beta}$ is established with φ an homomorphism and $\mathbb{C}$ is the set of $\Re$ endowed with the properties of $-a =_A a$; hence both maps are the same; hence the holomorphic property define before maintains.

On this context the dimension of the real space remains $\Re^{2n}$

Now lets define the property to be a Kahler manifold

3 Kähler Manifold

Def. Kähler manifold. In mathematics and specially differential geometry is a manifold with three mutually compatible structures: a complex structure, a Riemannian structure, and a symplectic structure.

4 Kähler Manifold is a Symplectic Manifold

Def. A symplectic form on a smooth manifold M is a closed non-degenerate differential 2-form ω. A symplectic manifold is a pair (M, ω) where M is a smooth manifold and ω is a symplectic form. Assigning a symplectic form to M is referred to as giving M a symplectic structure.

Def. Let M be a smooth manifold. A smooth differential form of degree k is a smooth section of the k-th exterior power of the cotangent bundle of M. The set of all differential k-forms on a manifold M is a vector space, often denoted $\Omega^k(M)$.

Def. In the mathematical field of topology, a section (or cross section) of a fiber bundle E is a continuous right inverse of the projection function π. In other words, if E is a fiber bundle over a base space, B:

$\pi : E \to B$

then a section of that fiber bundle is a continuous map,

$\sigma : B \to E$

such that

$\pi(\sigma(x)) = x$ for all $x \in B$

Def. The k-th exterior power of V, denoted $\bigwedge^k(V)$, is the vector subspace of $\bigwedge(V)$ spanned by elements of the form

$x_1 \wedge x_2 \wedge ... \wedge x_k,\ x_i \in V, \quad i = 1, 2, ..., k$

If $\alpha \in \bigwedge^k(V)$, then α is said to be a k-vector.

This property only is defined for k = 2 (see next proof)

Def. The exterior algebra $\bigwedge(V)$ of a vector space V over a field K is defined as the quotient algebra of the tensor algebra $T(V)$, where

$T(V) = \bigoplus_{k=0}^{\infty} T^k V = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus ...,$

by the two-sided ideal I generated by all elements of the form $x \otimes x$ such that $x \in V$. Symbolically,

$\bigwedge(V) := T(V)/I$

The exterior product $\wedge$ of two elements of $\bigwedge(V)$ is defined by

$\alpha \wedge \beta = (\alpha \otimes \beta)(\text{mod} \quad I)$

Def (Quotient algebra). In mathematics, a quotient algebra is the result of partitioning the elements of an algebraic structure using a congruence relation.

An associative algebra A over a commutative ring R is a ring itself. If I is an ideal in A (closed under R - multiplication), then A/I inherits the structure of an algebra over R and is the quotient algebra.

Def. An associative algebra A is a ring together with a ring homomorphism from R to the center of A.

Def. Tensor Algebra. In mathematics, tensor algebra is a construction of an associative algebra $(T(V), +, \otimes)$ starting from a vector space $(V, \mathbb{K}, +)$ over the body $\mathbb{K}$.

The construction of $T(V)$ is a direct sum of graduated parts T^k for $k = 0, 1, 2, ...$

$$T(V) = \bigoplus_k T^k(V) = T^0(V) \oplus T^1(V) \oplus T^2(V) \oplus ...$$

Where T^k is the tensor product of V with itself k times:

$$T^k(V) := \overbrace{V \otimes ... \otimes V}^{k}$$

and $T^0(V) = \mathbb{K}$ is a one-dimensional vector space. The multiplication function in T^i and T^j maps T^{i+j} and is the natural juxtaposition of pure tensors, enlarged by bilinearity. That is, tensor algebra is representative of algebras with covariant tensors that are formed of V and of any range. To have the complete algebra of tensors, contravariants as well as covariants, it must be taken $T(W)$ where W is the direct sum of V and of their dual space - this will consist of all tensors T_I^J with the upper indices J and lower indices I, in classical notation.

Def. The tensor product $V \otimes_K W$ of two vector spaces V and W on a field K have a formal definition by the method of generators and relations (usually denoted as $V \otimes W$ when the underlying field K is implied). To construct it, we start with the set of ordered pairs of the Cartesian product $V \times W$. For the purpose of this construction, consider this product as a set rather than a vector space. The free vector space F over $V \times W$ is defined by taking the vector space in which the elements of $V \times W$ are a base. Written in set theoretical notation,

$$F(V \times W) = \Big\{ \sum_{i=1}^{n} \alpha_i e_{(v_i, w_i)} \Big| n \in \mathbb{N}, \alpha_i \in K, (v_i, w_i) \in V \times W \Big\},$$

where the symbol $e_{(V,W)}$ is used to emphasize that they are taken as linearly independent by definition for different $(V, W) \in V \times W$.

The tensor product arises from the definition of the following equivalence relations in $F(V \times W)$:

$$e_{(v_1+v_2,w)} \sim e_{(v_1,w)} + e_{(v_2,w)}$$

$$e_{(v,w_1+w_2)} \sim e_{(v,w_1)} + e_{(v,w_2)}$$

$$ce_{(v,w)} \sim e_{(cv,w)} \sim e_{(v,cw)}$$

where v, $v1$, $v2$ are vectors of V, while w, $w1$ and $w2$ are vectors of W, and c arises from the field K.

Denoting by R the space generated by these four equivalence relations, the tensor product of two vector spaces V and W is then the quotient space

$$V \otimes W = F(V \times W)/R$$

It is also called the tensor product space of V and W and is a vector space (which can be verified directly by looking at the axioms of vector space). The two-element tensor product v and w is the equivalence class $(e_{v,w} + R)$ of $e_{(v,w)}$ in $V \otimes W$. The equivalence class of (v, w) is called the tensor and is denoted by $v \otimes w$. By construction, only as many identities between the tensors, and the sum of tensors, can be demonstrated from the relations used.

Taking the vector space generated by $W \times V$ and apply the multilinear relationships detailed above. With this notation, the four equivalence relations take the form of equalities in the tensor-product space:

*) $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$

*) $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$

*) $cv \otimes w = v \otimes cw = c(v \otimes w)$

Given bases for V and W, the set of tensor products of the base vectors, one of V and one of W, form a basis for $V \otimes W$.

The dimension of space is therefore given by the product mn of the dimensions of V and W.

Proof.
Lets consider the definition of the $F(X \times Y)$ set

$$F(X \times Y) = \Big\{ \sum_{i=1}^{n} \alpha_i e_{(x_i,y_i)} \Big| n \in \mathbb{N}, \alpha_i \in K, (x_i, y_i) \in X \times Y \Big\},$$

with

$e_{(x_i,y_i)}$ a base

and

$(x_i, y_i) \in X \times Y$

Lets take into account that equivalence relation establish that $e_{(x_i,y_i)}$ is linear (due to addition and multiplication by an scalar)

Now consider

$$\det \begin{vmatrix} x_i & y_i \\ -e_j & e_i \end{vmatrix} = x_i e_i + y_i e_j = c$$

Taking

$x_i a + y_i b = \alpha$

$(-e_j)a + (e_i)b = \beta$

like two linear equations with $x_i \in X$ and $y_i \in Y$ within $e_i \in X$ and $e_j \in Y$ on same spaces

with a and b variables and α and $\beta \in \Re$

Like the symbol $e_{(X,Y)}$ is used to emphasize that they are taken as linearly independent by definition for different $(X, Y) \in X \times Y$ hence with

$c = x_i e_i + y_i e_j$

defining the structure of a vector space with scalar product hence both terms $e_{(x_i,y_i)}$ and $c = x_i e_i + y_i e_j$ have the same properties.

Hence

$$e_{(x_i,y_i)} := x_i e_i + y_i e_j$$

with e_i, e_j the entries of a base on X and Y

Hence the space of the equivalence relations is

$$R = R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5$$

where

$$R_1 = \{c \quad | \quad e_{(x_1+x_2,y)} := (x_1 + x_2)e_i + ye_j = x_1e_i + x_2e_i + ye_j\} \quad ;$$

$$R_2 = \{c \quad | \quad e_{(x,y_1+y_2)} := xe_i + (y_1 + y_2)e_j = xe_i + y_1e_j + y_2e_j\} \quad ;$$

$$R_3 = \{c \quad | \quad ke_{(x,y)} := k(xe_i + y_ej) = kxe_i + kye_j\} \quad ;$$

$$R_4 = \{c \quad | \quad e_{(kx,y)} := kxe_i + ye_j\} \quad ;$$

$$R_5 = \{c \quad | \quad e_{(x,ky)} := xe_i + kye_j\}$$

Hence $c \in R$ with $(x, y) \in X \times Y$ is such that

$$c = \sum_{s=1}^{n} \alpha_s(x_se_i + y_se_j)$$

where index i of sum has been rename to let e_i and e_j represent that the values of such entries is on $\Re$ (e.g. e_i, e_j could be taken in the case of unit vectors as entries of value 1)

Def. In linear algebra, the quotient of a vector space V by a subspace U is a vector space obtained by "collapsing" U to zero. The space obtained is called a quotient space and is denoted V/U.

The quotient space V/U is then defined as $V/\sim$, the set of all equivalence classes induced by $\sim$ on U.

Proof.

With $F(X \times Y)/R$ the quotient space hence the tensor product space is defined as

$$X \otimes Y := F(X \times Y)/R$$

or

$$X \otimes Y = R$$

with $X, Y \subseteq \Re$ the dual space of $X \times Y$ and $F(\vec{A}) = A\vec{v}$ the vector function; where $A = \alpha q(t)$; with α an scalar and $q(t)$ the linear functional; is given by ref[1]

$$x_i = q(t)x_i' \quad ;$$

$$y_i = q(t)y_i' \quad ;$$

such that the distance between both points is

$$d[(x_i, y_i), q(t)(x_i', y_i')] = q(t)$$

such that the dual space is

$$SB^* = \{q(t) \in Q(t) \mid R(\theta)\vec{v} = q(t)R(\theta)\vec{v}\}$$

with $\vec{v} = (x, y)$ and $\vec{v'} = (x', y')$

where

$$x \neq x'$$

and

$$y \neq y'$$

with $\vec{v} \in B$ (base space).

Hence taking into consideration next definition

Def. For non-negative integers r and s a type (r, s) tensor on a vector space V is an element of:

$$T_s^r(V) = \underbrace{V \otimes ... \otimes V}_{r} \otimes \underbrace{V^* \otimes ... \otimes V^*}_{s} = V^{\otimes r} \otimes (V^*)^{\otimes r}$$

Here V^* is the dual vector space (which consist of all linear maps f from V to the ground field K)

Proof.

Hence the tensor is

$$T_s^r(V) = \underbrace{X \otimes ... \otimes X}_{r} \otimes \underbrace{X^* \otimes ... \otimes X^*}_{s} = X^{\otimes r} \otimes (X^*)^{\otimes r}$$

with the dual space and vector space defined early

Now lets define the exterior algebra $\bigwedge^2(X)$

Note 1: With this definition

$$a \otimes b := \begin{cases} (a, b) \in X \times X \quad ; \\ ae_i + be_j = c \in R = T^2(X) \end{cases}$$

Note 2: $a \otimes b \neq c$ and only $c \in T^2(X)$

Note 3:

$$\underbrace{X \otimes ... \otimes X}_{s} := \begin{cases} \underbrace{\underbrace{x \times ... \times x}_{s}, ..., \underbrace{x \times ... \times x}_{s}}_{k-1} \in \underbrace{X \times ... \times X}_{s} \\ detA = c \in R = T^k(X) \\ \text{such that} \quad A \quad \text{is} \quad k \times k \end{cases}$$

Note 4: To define the determinant of the matrix A of $s \times s$ entries it is necessary to take $k-1$ s-tuples; i.e. the tensor product $\underbrace{X \otimes ... \otimes X}_{s}$ that defines the set $T^s(X)$ need s-1 tuples such that the tuple of s entries is on X^s; i.e.

$$\underbrace{(x, .., x)}_{s} \in \underbrace{X \times ... \times X}_{s}$$

Note 5: Must be evident that each x by tuple could be different; here such fact is not indicated with any symbol.

*) $T^2(X)$ is an associative algebra

Def. A ring is a set R equipped with two binary operations, i.e. operations combining any two elements of the ring to a third. They are called addition and multiplication.

*) R is an abelian group under addition

With

$$c \in T^2(X) \quad ;$$

The term c is defined with $(x, y) \in X \times Y$ like

$xe_i + ye_j = c \in R$

Where c is obtained since the determinant of the matrix of the system of the next linear equations

$$xa + yb = \alpha \quad ;$$

$$-(e_j)a + e_i b = \beta \quad ;$$

Like $c \in \Re$ and like any subset of $\Re$ is a commutative group hence R is an abelian group under addition.

*) R is a monoid under multiplication

Like $c = 1 \in R$ (identity element) and like $c_1 \cdot c_2 \in R$ for all $c_1, c_2 \in R$ (closure) hence R is a monoid under multiplication

*) Multiplication distributes over addition

With addition and multiplication defined in arithmetic sense hence like distributivity is defined on reals hence R is multiplicative distributive over addition.

Therefore R is a ring

Like multiplication is commutative hence R is a commutative ring

*) φ is a ring homomorphism

Proof.

Lets establish next definition for φ function

$$\varphi : R \to Z(A) \quad ;$$

where

$Z(A) := Im[\varphi(R)]$

If φ is

*) φ is bijective

hence if $\varphi(c_1) = \varphi(c_2)$ therefore $c_1 = c_2$

and

$\varphi(c) = \gamma$ therefore $\forall \gamma \in Im[\varphi(R)] \quad ; \quad \exists c \in R$

*) φ is continuous (between topological spaces R and $Z(A)$)

with $B_r(c_p) = \{c_a \in R \quad | \quad d(c_a, c_p) < r\}$

therefore R is a topological space

if

$R \in \Re$

and

$\varphi(R) \in \Re$

hence with

$$B_r(\varphi(c_p)) = \{\varphi(c_a) \in Z(A) \mid d(\varphi(c_a), \varphi(c_p)) < r\}$$

therefore $Z(A)$ is a topological space

hence

$$\varphi^{-1}(Z(A)) = \{c \in R \mid \varphi(c) \in Z(A)\}$$

*) φ^{-1} is continuous (between topological spaces)

$$\varphi(R) = \{\ \varphi(c) \in Z(A) \mid c \in R\}$$

hence φ is an homomorphism

*) $Im[\varphi(R)]$ is the center of a ring

Def. In algebra, the center of a ring R is the sub-ring consisting of the elements x such that $xy = yx$ for all elements y in R. Also is a commutative ring denoted $Z(R)$

Proof.

If

$$1, 0 \in \varphi(R)$$

and addition and multiplication have arithmetic definition on $\varphi(R)$ hence

+) $Z(A) := Im[\varphi(R)]$ is an abelian group under addition

+) $Z(A) := Im[\varphi(R)]$ is a monoid under multiplication

+) multiplication is distributive with respect to addition on $Z(A) := Im[\varphi(R)]$

such that

$Z(A) := Im[\varphi(R)] \subseteq \Re$

where it is considered that addition and multiplication in arithmetic sense is defined on $\Re$ and hence is a ring.

with

$x \in Im[\varphi(R)]$ and $y \in \Re$

Like multiplication is commutative in $Im[\varphi(R)]$ and $\Re$ hence

$$xy = yx$$

hence

$Im[\varphi(R)]$ is a center of the ring $\Re$

Hence $T^2(X)$ is an associative algebra

*) $\bigwedge^2(X) := T^2(X)/I$ is a quotient algebra

Def. Given a ring R, a left ideal is a subset I of R that is a subgroup of the additive group of R that "absorbs multiplication from the left by elements of R" ; that is, I is a left ideal if it satisfies the following two conditions:

1. $(I, +)$ is a subgroup $(R, +)$

2. For every $r \in R$ and every $x \in I$, the product rx is in I

A right ideal is defined similarly, with the condition $rx \in I$ replaced by $xr \in I$. A two-sided ideal is a left ideal that is also a right ideal.

If the ring is commutative, the definitions of left, right, and two-sided ideal coincide, and one talks simply of an ideal.

Proof.

1)

Let $I \subseteq R$ such that

$0 \in I$

and

$\forall c \in I; \quad \exists - c \in I$

Hence I is a group

Like R is a group and $I \subseteq R$

$I \leq G$

2)

If $c = rx$ (or xr)

hence

$rx \in I \subseteq R$

else with $r_1, r_2, r \in R$ like

$r = r_1 \cdot r_2 \in R$

and

$r = r_1 + r_2 \in R$

and with $x, x_1, x_2 \in I$

$x = x_1 + x_2 \in I$

Hence I is a right ideal

Like $+, \cdot$ are commutative for reals $\Re$

Hence I is a left ideal

Hence I is an ideal

Def. Formal quotient ring construction: Given a ring R and a two-sided ideal I in R, we may define an equivalence relation $\sim$ on R as follows: $a \sim b$ if and only if $a - b$ is in I.

Proof.

$\supseteq$

Let be

*)

$r_a x_a - r_a x_a \in I$

(or $x_a r_a - x_a r_a \in I$)

hence

$0 \in I$

hence

$0 \in R$

therefore

$r_a x_a - r_a x_a \in R$

therefore

$r_a x_a \sim r_a x_a$ is reflexive on R

**)

Let be

$r_a x_a - r_b x_b \in I \subseteq R$

(or $x_a r_a - x_b r_b \in I \subseteq R$)

hence

$r_b x_b - r_a x_a \in I$

hence

$r_b x_b - r_a x_a \in R$

hence

$r_a x_a \sim r_b x_b$ is symmetric $\in R$

***)

Let be

$r_a x_a - r_b x_b \in I \subseteq R \quad ;$

$r_b x_b - r_c x_c \in I \subseteq R$

(or $x_a r_a - x_b r_b \in I \subseteq R$

$x_b r_b - x_c r_c \in I \subseteq R$)

hence

$r_a x_a - r_b x_b + r_b x_b - r_c x_c = r_a x_a - r_c x_c \in I \subseteq R$

hence

$r_a x_a \sim r_c x_c$ is transitive $\in R$

therefore $r_a x_a \sim r_b x_b := r_a x_a - r_b x_b \in I$ is an equivalence relation on R

$\subseteq$

Like

$r_a x_a \sim r_b x_b \in R$ is an equivalence relation

hence

$r_a x_a - r_b x_b \in I$

therefore $\bigwedge^2(V) := T(V)^2/I$ is a quotient algebra

such that the exterior product $\wedge$ of two elements of $\bigwedge^2(V)$ is

$$\alpha \wedge \beta := \begin{cases} (a,b) \in X \times X \subseteq E \quad ; \\ a e_i + b e_j = c \in R = T^2(X) \quad ; \\ c_1 - c_2 \in I \quad \text{such that} \quad c = xr(\mathrm{rx}) \end{cases}$$

where E is the total space and here $E = \mathbb{C}$

Note that a k vector is a c defined with k-1 exterior products for each exterior power $\bigwedge^k(V)$ like was defined before.

*) Cross section

Def. If E is a fiber bundle over a base space, B:

$\pi : E \to B$

then a section of that fiber bundle is a continuous map,

$\sigma : B \to E$

such that

$\pi(\sigma(x)) = x$ for all $x \in B$

Proof.

Def. A map $\phi : X \to Y$ is continuous iff the preimage of any open set is open.

Proof.

Taking

$$\sigma : B \to E \quad ;$$

such that

$$\sigma(B_r(x'_1, x'_2)) = B_r(x_1, x_2) \quad ;$$

where $B_r(x'_1, x'_2) \subseteq B$ and $B_r(x_1, x_2) \subseteq E$ are open sets

with

$$E = \mathbb{C} \quad ;$$

and

$B = \{(x'_1, x'_2) \in \mathbb{C} \quad | \quad (x'_1, x'_2) \in \bigcup_{i \in Z^+ \cup Z^-} S_p M_i \quad \wedge \quad (x'_1, x'_2) \in \bigcap_{i \in Z^+ \cup Z^-} S_p M_i\} \quad ;$

like was established early

i.e.

$$\sigma(x'_1, x'_2) = (x_1, x_2) \in E$$

Hence σ is continuous

Remember that

$\pi = \{(x_1, x_2) \in \mathbb{C} \quad | \quad (x_1, x_2) \in C \wedge (x'_1, x'_2) \in S_pM \wedge (x'_1, x'_2) \in B \wedge (x_1 \neq x'_1 \quad ; \quad x_2 \neq x'_2)\}$

Hence if π is applied on σ such that

$\pi(\sigma(x'_1, x'_2)) : E \to B$

with $E = \mathbb{C}$

and

$$\sigma(x'_1, x'_2) \in E$$

hence

$$\pi(\sigma(x'_1, x'_2)) = (x'_1, x'_2) \in B$$

hence π have same definition as early; i.e. the section σ is a continuous right inverse of the projection function π

*) Smooth differential form of degree k is a smooth section of the k-th exterior power of the cotangent bundle

Lets take the set

$$C_{(h_1,h_2)} = \{(x, y) \in \mathbb{C} \quad | \quad (x + h_1)^2 + (y + h_2)^2 = r^2\}$$

to define a circle that has center out of the origin.

Now consider next set

$$L = \{(x + h_1, y + h_2) \in C_{(h_1,h_2)} \quad | \quad (y + h_2) = m(x + h_1) + b\}$$

Taking into consideration ref[5] and translation equality

$$(x, y) =_{T(h_1,h_2)} (x + h_1, y + h_2)$$

Hence a circle define by $h_1 = h_2 = 0$ is the same that a circle out of the origin; and the same for the straight line.

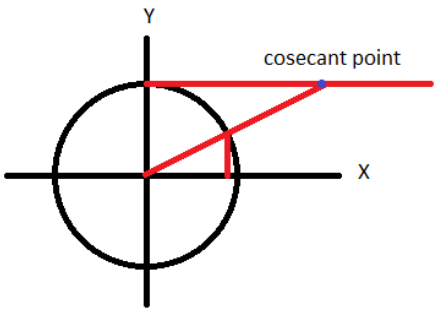
Now lets define a straight line through two points $(x_1, x_2) \in B$ and $(x'_1, x'_2) \in SB$

$$L_{\text{secant}_{[(x_1,x_2),(x'_1,x'_2)]}} = \{(x^*_1, x^*_2) \in C_{(0,0)} \subset E \quad | \quad x^*_2 = (\frac{x'_2 - x_2}{x'_1 - x_1})(x^*_1 - x_1) + x_2 \quad ;$$

$$d = \sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2} \quad ;$$

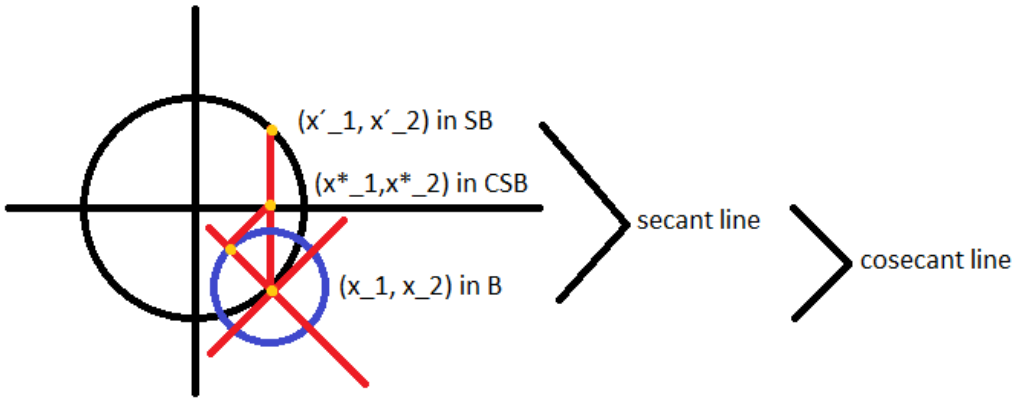
$$(x_1, x_2) \in B \quad \text{and} \quad (x'_1, x'_2) \in SB\}$$

Remember that a cosecant point can be defined as



cosecant point

Hence taking into consideration the definition of secant line on the bundle [see next scheme]



Cosecant Bundle CSB; SB, and B

The point (x_1^*, x_2^*) is a cosecant point of the line L while the set

$$CSB_{(0,0)} = \{(x_1^*, x_2^*) \in L_{\text{secant}} \quad |$$

1) $(x_a, x_b) \in B_r(x_a, x_b)$i.e. is the center of such circle ;

$$2)(x_1, x_2) \in B_r(x_a, x_b) \quad ;$$

$$3)\sqrt{(x_a - x_1)^2 + (x_b - x_2)^2} = r \quad ;$$

$$4)\sqrt{(x_a - x_1^*)^2 + (x_b - x_2^*)^2} = p \quad ;$$

$$5) r^2 + p^2 = q^2 \quad ;$$

$$6) \sqrt{(x_1^* - x_1)^2 + (x_2^* - x_2)^2} = q \quad ;$$

such that $\forall (x_1, x_2)$ and $(x_1', x_2') \in L_{secant} \subset C_{(0,0)}; (x_1^*, x_2^*)$ is a cosecant point of $\mathrm{L}_{secant}\}$

defining the set of cosecant points for the circle $C_{(0,0)}$.

Later by translation $=_T$ of the point (x_a, x_b) center of the circle C

$CSB_{(h_1,h_2)} = \{(x_1^*, x_2^*) \in L_{secant} \quad | \quad 1), 2), 3), 4), 5)$ and $5)$ are fulfilled such that

$\forall (x_1, x_2)$ and $(x_1', x_2') \in C_{(h_1,h_2)}; (x_1^*, x_2^*)$ is cosecant point of $L_{\mathrm{secant}_{[(x_1,x_2),(x_1',x_2')]}}\}$

i.e. such set is the set of all cosecant points on $\mathbb{C}$

Like such line L and cosecant point are defined for all $\Re^2$

Hence for every h_1, h_2, x_1 and x_2 defining an open set on $\mathbb{C}$;

$CSB_{(h_1,h_2)} = \mathbb{C}$ ($\Re^2$ endowed with $-a =_A a$ to have the properties of $\mathbb{C}$)

Now take into account the circle C' with center on (x_a, x_b)

$C'_{(x_a,x_b)} = \{(x_1, x_2) \in E \quad | \quad d[(x_a, x_b), (x_1^*, x_2^*)] = r \quad \mathrm{and} (x_1^*, x_2^*) \in CSB_{(h_1,h_2)}\}$

where

$$d[(x_a, x_b), (x_1, x_2)] < r$$

defines again that each $(x_1^*.x_2^*)$ on the perimeter of C' is on B; i.e.

$$(x_1^*, x_2^*) \in B$$

hence the fiber section on such point can be defined as

$$(x_1^*, x_2^*) \in C' \quad ;$$

$$\pi(\sigma(x_1^*, x_2^*)) = (x_1^*, x_2^*) \in B$$

and the cosecant bundle is equal to the bundle of $E = \mathbb{C}$ and $B = SB$; where the point used to define the radius of the circle can be defined as an element of $CSB_{(h_1,h_2)}$ i.e. the cosecant space defined with the circle of center (h_1, h_2).

hence ω := *form of degree* $k = 2$ is defined with

the smooth section

$$\sigma : CSB_{(h_1,h_2)} \to E$$

of the *2 - exterior power* of the cotangent bundle

$$\alpha \wedge \beta := \begin{cases} (a,b) \in X \times X \subseteq CSB_{(h_1,h_2)} & ; \\ ae_i + be_j = c \in R = T^2(X) & ; \\ c_1 - c_2 \in I \quad \text{such that} \quad c = xr(\text{rx}) \end{cases}$$

*) Non Degenerate. Here, non-degenerate means that for every point $p \in M$, the skew-symmetric pairing on the tangent space T_pM defined by ω is non-degenerate. That is to say, if there exists an $X \in T_pM$ such that $\omega(X, Y) = 0$ for all $Y \in T_pM$, then $X = 0$. Since in odd dimensions, skew-symmetric matrices are always singular, the requirement that ω be nondegenerate implies that M has an even dimension.

Def. Let V a vector space over a body k

A skew-symmetric pairing is a bilinear map

$$< \cdot, \cdot >: V \times V \to k \quad ;$$

such that

$$< v, w >= - < w, v > \quad ; \forall v, w \in V$$

Example: On $\Re^2$ defines a skew-symmetric pairing:

$$< (x_1, y_1), (x_2.y_2) >= x_1 y_2 - x_2 y_1$$

Proof.

Taking the tensor product

$$c = ae_i + be_j \in R = T^2(x)$$

of

$$a \wedge b \in \bigwedge^k (X)$$

such that

$$(a, b) \in Range(\sigma)$$

where

$$Range(\sigma) = \{\sigma(x_1, x_2) \in E \quad | \quad (x_1, x_2) \in CSB_{(h_1,h_2)} \text{ and } (x_1, x_2) \in S_p M \text{ of } CSB_{(h_1,h_2)}\}$$

with

$\sigma :=$ the fiber section

if

$$detA = c = 0$$

where

$$A = \begin{vmatrix} a & b \\ -e_j & e_i \end{vmatrix}$$

hence

$a, b = 0 \quad ; \forall e_j, e_i \in \Re$

i.e.

e_j and e_i can have any value; i.e. the base can have any value on its entries.

therefore the 2-form is non degenerate.

Else like

$$detA =< (a, b), (-e_j, e_i) >$$

therefore the skew symmetric pairing is

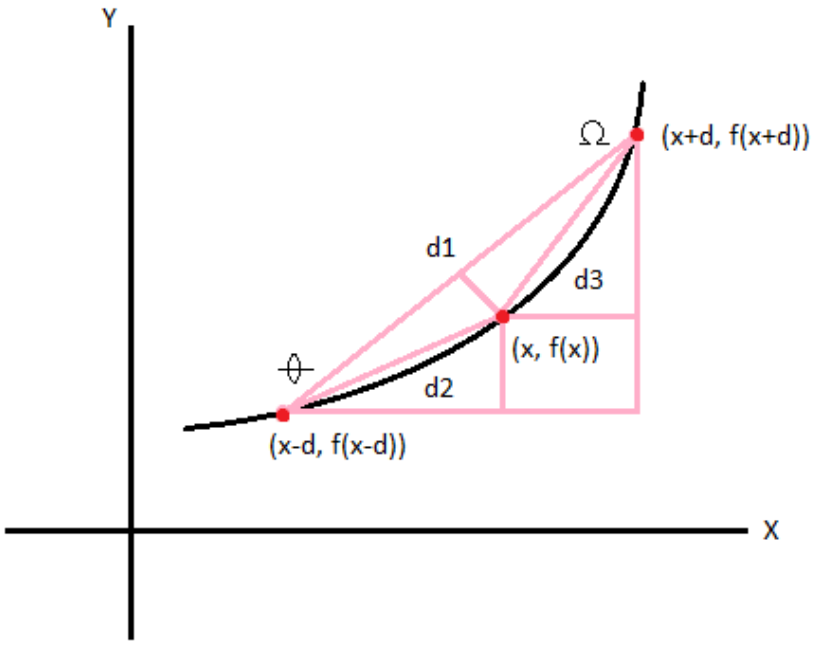
$$(a, b) \otimes (-e_j, e_i) =< (a, b), (-e_j, e_i) >= detA$$

*) Closed. The closed condition means that the exterior derivative of ω vanishes.

Def. The derivative of a function at a point **a** is the slope **m** of the tangent line to the curve at that point and also the equation of a straight line with pendant $m = f'(a)$

$$y - f(a) = f'(a)(x - a)$$

Proof.



Scheme of tangent line

Lets consider the point $(x_1^*, x_2^*) \in CSB_{(h_1,h_2)}$ such that

$$C'_{(a,b)} = \{ \quad (x_1, x_2) \in E \quad | \quad d[(a,b),(x_1^*, x_2^*)] = r \quad \}$$

and consider the definition of derivative on a point

Now consider the point $(x_1^*, x_2^*) \in CSB_{(h_1,h_2)}$ and the definition of a derivative given at the beginning of this text for a holomorphic function.

Lets establish next relation for x_1^* and δ

$$x_1^* - \delta < x_1^* < x_1^* + \delta$$

Now consider a function $f(x)$ where

$$x_1^* - \delta, \quad x_1^*, \quad x_1^* + \delta \in \text{Domain [f(x)]}$$

and each $f(x)$ establish the value of the coordinate x_2^* in each case

Later, substituting each value in $f(x)$ establish a trichotomy relation.

Now lets consider next condition

$$f(x_1^* - \delta) < f(x_1^*) < f(x_1^* + \delta);$$

and also consider the distance by pairs

$$d_1 = d[(x_1^* - \delta, f(x_1^* - \delta)), (x_1^* + \delta, f(x_1^* + \delta))]$$

$$d_2 = d[(x_1^*, f(x_1^*)), (x_1^* - \delta, f(x_1^* - \delta))]$$

$$d_3 = d[(x_1^* + \delta, f(x_1^* + \delta)), (x_1^*, f(x_1^*))]$$

Now consider the triangle rectangle form by $x_1^* - \delta$ and $x_1^* + \delta$

$$(x_1^* + \delta) - (x_1^* - \delta) = d_1 cos\theta \quad ;$$

$$f(x_1^* + \delta) - f(x_1^* - \delta) = d_1 sin\theta$$

clearing θ

$$\theta = arccos(\frac{(x_1^* + \delta) - (x_1^* - \delta)}{d_1}) \quad ;$$

$$\theta = arcsin(\frac{f(x_1^* + \delta) - f(x_1^* - \delta)}{d_1})$$

Now consider the triangle form by x_1^* and $x_1^* - \delta$

$$x_1^* - (x_1^* - \delta) = d_2 cos\phi \quad ;$$

$$f(x_1^*) - f(x_1^* - \delta) = d_2 sin\phi$$

clearing ϕ

$$\phi = arccos(\frac{x_1^* - (x_1^* - \delta)}{d_2});$$

$$\phi = arcsin(\frac{f(x_1^*) - f(x_1^* - \delta)}{d_2})$$

Hence, the angle between d_1 and d_2 is

$$\angle d_1 d_2 = \theta - \phi$$

Now consider the triangle form by $x_1^* + \delta$ and x_1^*

$$(x_1^* + \delta) - x_1^* = d_3 cos\Omega \quad ;$$

$$f(x_1^* + \delta) - f(x_1^*) = d_3 sin\Omega$$

clearing Ω

$$\Omega = arccos(\frac{(x_1^* + \delta) - x_1^*}{d_3});$$

$$\Omega = arcsin(\frac{f(x_1^* + \delta) - f(x_1^*)}{d_3})$$

Like

$$\Omega + 90 + \eta = 180$$

Hence

$$\eta = 180 - 90 - \Omega$$

While for triangle $\triangle d_1 d_2 d_3$ and congruence of corresponding angles for θ

$$\theta + 90 + \eta + \angle d_1 d_3 = 180$$

Clearing $\angle d_1 d_3$

$$\angle d_1 d_3 = 180 - 90 - \eta - \theta$$

Now lets establish a parallel line to d_1 through $(x_1^*, f(x_1^*))$

Taking the triangle form by d_1, d_2 and d_3 considering 90 angles on its height h hence

$$d_1 = d_2 cos(\angle d_1 d_2) + d_3 cos(\angle d_1 d_3)$$

and

$$d_2 sin(\angle d_1 d_2) = d_3 sin(\angle d_1 d_3) = h$$

*) By congruence of alternate angles; hence

$\angle d_1 d_3$ must be equal to the angle of a parallel to d_1 through $(x_1^*, f(x_1^*))$ to the point that is parallel to $(x_1^* + \delta, f(x_1^* + \delta))$

Hence

the length of a parallel to d_1 through $(x_1^*, f(x_1^*))$ is $d_3 cos(\angle d_1 d_3)$

*) By congruence of corresponding angles; hence

$\angle \theta$ must be equal to the angle of a parallel to d_1 through $(x_1^*, f(x_1^*))$ to the point that is parallel to $(x_1^* + \delta, f(x_1^* + \delta))$

Hence the x-coordinate of the point parallel to $(x_1^* + \delta, f(x_1^* + \delta))$

$$\text{is} \quad d_3 cos(\angle d_1 d_3) cos(\theta) + x_1^* = x_{\|_{\delta+}}$$

Hence the y-coordinate of the point parallel to $(x_1^* + \delta, f(x_1^* + \delta))$

$$f(x_1^*) + d_3 cos(\angle d_1 d_3) sin(\theta) = y_{\|_{\delta+}}$$

*) By congruence of alternate angles; hence

$\angle d_1 d_2$ must be equal to the angle of a parallel to d_1 through $(x_1^*, f(x_1^*))$ to the point that is parallel to $(x_1^* - \delta, f(x_1^* - \delta))$

Hence

the length of a parallel to d_1 through $(x_1^*, f(x_1^*))$ is $d_2 cos(\angle d_1 d_2)$

*) By congruence of corresponding angles; hence

$\angle \theta$ must be equal to the angle of a parallel to d_1 through $(x_1^*, f(x_1^*))$ to the point that is parallel to $(x_1^* - \delta, f(x_1^* - \delta))$

Hence the x-coordinate of the point parallel to $(x_1^* - \delta, f(x_1^* - \delta))$

$$\text{is} \quad x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta) = x_{\|_{\delta-}}$$

Hence the y-coordinate of the point parallel to $(x_1^* - \delta, f(x_1^* - \delta))$

$$\text{is} \quad f(x_1^*) - d_2 cos(\angle d_1 d_2) sin(\theta) = y_{\|_{\delta-}}$$

Now lets obtain the equation of the straight line between two points

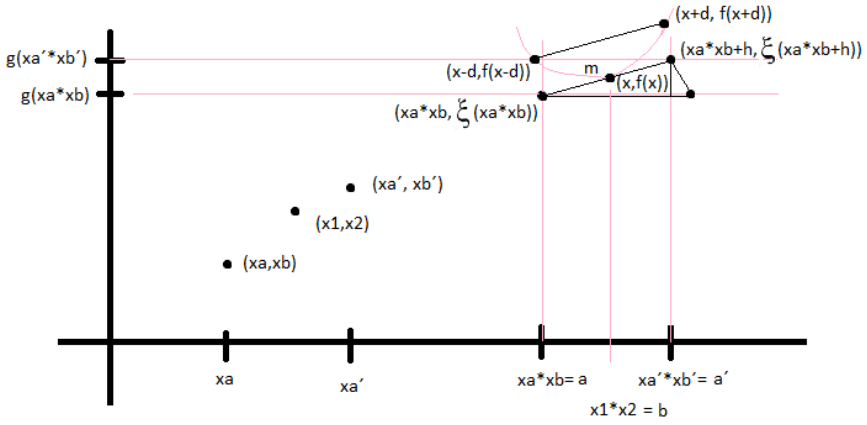
with m the y-intercept with equation

$$m_{\|} = \frac{(f(x_1^*) + d_3 cos(\angle d_1 d_3) sin(\theta)) - (f(x_1^*) - d_2 cos(\angle d_1 d_2) sin(\theta))}{(x_1^* + d_3 cos(\angle d_1 d_3) cos(\theta)) - (x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta))}$$

Hence

$$y = m_{\parallel}(x - x_1^*) + f(x_1^*)$$

is the equation of the straight line through $(x_1^*, f(x_1^*))$



Scheme of tangent line and $f(x_1, x_2) := g(Prod(x_1, x_2))$

Like both points are part of the same tangent line hence

*) Lets establish each coordinate in terms of a dependent and an independent variable; taking a ξ function

$$\xi(x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta)) = f(x_1^*) - d_2 cos(\angle d_1 d_2) sin(\theta)$$

$$\xi(d_3 cos(\angle d_1 d_3) cos(\theta) + x_1^*) = d_3 cos(\angle d_1 d_3) sin(\theta) + f(x_1^*)$$

*) Now lets establish the relation between coordinates like function of $h :=$ the distance between points in X axis.

$$x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta) + h = d_3 cos(\angle d_1 d_3) cos(\theta) + x_1^*$$

$$\xi(x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta) + h) = d_3 cos(\angle d_1 d_3) sin(\theta) + f(x_1^*)$$

also

$$h = 2 * \delta$$

and

$$\xi = m_{\|}$$

Lets note that $f(x)$ can be expressed cleared and be established through any of its identities

$$1) \quad f(x_1^*) = \xi(d_3 cos(\angle d_1 d_3) cos(\theta) + x_1^*) - d_3 cos(\angle d_1 d_3) sin(\theta)$$

$$2) \quad f(x_1^*) = \xi(x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta)) + d_2 cos(\angle d_1 d_2) sin(\theta)$$

$$3) \quad f(x_1^*) = \xi(x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta) + h) - d_3 cos(\angle d_1 d_3) sin(\theta)$$

Now lets consider next property

* **Def. Multiplicative decomposition property in real numbers.** For any real number $r \in \Re$, there exist $a, b \in \Re$ such that $r = a \cdot b$. This decomposition is not unique and allows for infinitely many possible combinations.

Proof.

Remember that a function $\psi(x_1, x_2) \in \Re$ with $(x_1, x_2) \in \mathbb{C}$ can be defined as

$$\psi(x_1, x_2) := g(Prod(x_1, x_2))$$

Hence taking

$$x_{\|_{\delta -}} = x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta)$$

Hence decomposing in x_1 and x_2

$$x_1 * x_2 = d_2 * (\frac{x_1^*}{d_2} - cos(\angle d_1 d_2) cos(\theta))$$

$$Prod(x_1, x_2) = Prod(\quad d_2 \quad , \quad \frac{x_1^*}{d_2} - cos(\angle d_1 d_2) cos(\theta) \quad)$$

Multiplying by $m = \xi$

$$m * Prod(\quad d_2 \quad , \quad \frac{x_1^*}{d_2} - cos(\angle d_1 d_2) cos(\theta) \quad)$$
$$=$$
$$\xi(Prod(\quad d_2 \quad , \quad \frac{x_1^*}{d_2} - cos(\angle d_1 d_2) cos(\theta) \quad))$$

$$=$$

$$\xi(Prod(x_1, x_2))$$

$$=$$

$$y_{\parallel_{\delta -}}$$

Hence

$$\psi(x_1, x_2) := \xi(Prod(x_1, x_2))$$

*) Now lets obtain the y-intercept of the parallel line to d_1 with the last relation used to establish ξ function.

$$m_{L_\parallel} := \frac{\xi(x_1 * x_2 + h) - \xi(x_1 * x_2)}{(x_1 * x_2 + h) - (x_1 * x_2)} = tan(\theta)$$

where $tan(\theta)$ establish the tangent relation between the height and base of a triangle rectangle formed by the parallel line $L_\parallel$, $L_\parallel cos(\theta)$ and $L_\parallel sin(\theta)$; with θ the angle that establish the congruence between both parallel lines.

To be a parallel hence d_1 is such that

$$tan(\theta) = \frac{f(x+\delta) - f(x-\delta)}{(x+\delta) - (x-\delta)} =: m_{d_1}$$

Like the derivative of a function is established through

$$f = lim_{\delta \to 0} \frac{\xi(x_1 * x_2 + 2\delta) - \xi(x_1 * x_2)}{(x_1 * x_2 + 2\delta) - (x_1 * x_2)} = tan(\theta)$$

Hence the derivative of a function at a point $(x, f(x))$ is the tangent of the angle θ that establish the congruence between the two triangle rectangles that establish both parallel lines, where d_1 is a secant on the curve of the function $f(x)$.

Note that the parallel line criteria about $L_{\parallel}$ can not secure that the straight line is in fact a tangent line; even when $\delta \to 0$; because there is no a defined criteria to establish that $L_{\parallel}$ not pass through another point of the function between $x - \delta$ and $x + \delta$ besides x; instead, the meaning of tangent maintains with the meaning given above.

Also note that both definitions considered about the y-intercept; i.e., the y-intercept of a secant line d_1 and the y-intercept of $L_{\parallel}$; fulfilled the definition of "derivative"; however only one of them considers the point of derivation to establish its tangent relation through $tan(\theta)$ (i.e. the definition about line $L_{\parallel}$).

Now lets establish the relation of last definition about derivative an parallel line with the definition given at the beginning of this work for the derivative of a holomorphic function

Lets

$$sin(\theta) = \frac{f(x+\delta) - f(x-\delta)}{d[(x-\delta, f(x-\delta)), (x+\delta, f(x+\delta))]}$$

with

$$x - \delta + 2\delta = x + \delta$$

hence

$$sin(\theta) = \frac{f(x-\delta+h) - f(x-\delta)}{d[(x-\delta, f(x-\delta)), (x-\delta+h, f(x-\delta+h))]}$$

Now lets consider that

$$f(x) = d[(x, f(x)), (0,0)]sin(\alpha)$$

where

$$\alpha = arctan(\frac{f(x)}{x})$$

hence

$$sin(\theta) = \frac{d[(x-\delta+h, f(x-\delta+h)), (0,0)]sin(\alpha_{-\delta+h}) - d[(x-\delta, f(x-\delta)), (0,0)]sin(\alpha_{-\delta})}{d[(x-\delta, f(x-\delta)), (x-\delta+h, f(x-\delta+h))]}$$

hence

$$sin(\theta) = \frac{d_{-\delta+h}sin(\alpha_{-\delta+h}) - d_{-\delta}sin(\alpha_{-\delta})}{d[(x-\delta, f(x-\delta)), (x-\delta+h, f(x-\delta+h))]}$$

hence

$$sin(\theta) = \frac{Prod(d_{-\delta+h}, sin(\alpha_{-\delta+h})) - Prod(d_{-\delta}, sin(\alpha_{-\delta}))}{d[(x-\delta, f(x-\delta)), (x-\delta+h, f(x-\delta+h))]}$$

if $g = 1$ with $\Gamma(x_1, x_2) := g(Prod(x_1, x_2))$ hence

$$sin(\theta) = \Gamma' = \begin{matrix} lim_{d_{-\delta+h} \to (d_{-\delta})_0} \\ lim_{\alpha_{\delta+h} \to (\alpha_{-\delta})_0} \\ lim_{h \to 0} \end{matrix} \Big(\frac{\Gamma(d_{-\delta+h}, sin(\alpha_{-\delta+h})) - \Gamma(d_{-\delta}, sin(\alpha_{-\delta}))_0}{d[(x-\delta+h, f(x-\delta+h)), (x-\delta, x-\delta)_0]} \Big)$$

hence

$$sin(\theta) = \Gamma' = lim_{h \to 0} \Big(\frac{\Gamma(d_{-\delta+h}, sin(\alpha_{-\delta+h})) - \Gamma(d_{-\delta}, sin(\alpha_{-\delta}))}{d[(x-\delta+h, f(x-\delta+h)), (x-\delta, x-\delta)]} \Big)$$

while for the parallel line

$$sin(\theta) = \frac{\xi(x_1 * x_2 + h) - \xi(x_1 * x_2)}{d[(x - \delta, f(x - \delta)), (x - \delta + h, f(x - \delta + h))]}$$

where θ denotes the angle of the secant line between hypotenuse and the adjacent cathetus of the triangle rectangle that describes such line.

Note that holomorphic derivative definition allows to establish the relation between the points that establish the "tangent" derivative and the height of each point of the function in terms of the vector projection of the coordinates of such points.

Like before through congruence of triangle rectangles, the triangle of secant hypotenuse and parallel hypotenuse have same y-intercept this means that both lines, secant and tangent have same value of $sin(\theta)$

Note that when the y-intercept is zero, i.e. its an horizontal line hence

$$\angle d_1 d_2 = 0 \quad ; \angle d_1 d_3 = 0 \quad \text{and} \quad \theta = 0$$

*) For $(x_1^* + \delta, f(x_1^* + \delta))$

$$d_3 cos(\angle d_1 d_3) cos(\theta) + x_1^* =$$

$$d_3 cos(0) cos(0) + x_1^* =$$

$$d_3 + x_1^*$$

hence

$$d_3 = \delta$$

hence

$$\delta + x_1^* = x_{\|_{\delta +}}$$

and

$$f(x_1^*) + d_3 cos(0) sin(0) = f(x_1^*) = y_{\|_{\delta+}}$$

while for $(x_1^* - \delta, f(x_1^* - \delta))$

$$x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta) = x_1^* - d_2 cos(0) cos(0) = x_1^* - d_2 = x_{\|_{\delta-}}$$

hence

$$d_2 = \delta$$

hence

$$f(x_1^*) - d_2 cos(\angle d_1 d_2) sin(\theta) = f(x_1^*) - d_2 cos(\theta) sin(\theta) = f(x_1^*) = y_{\|_{\delta-}}$$

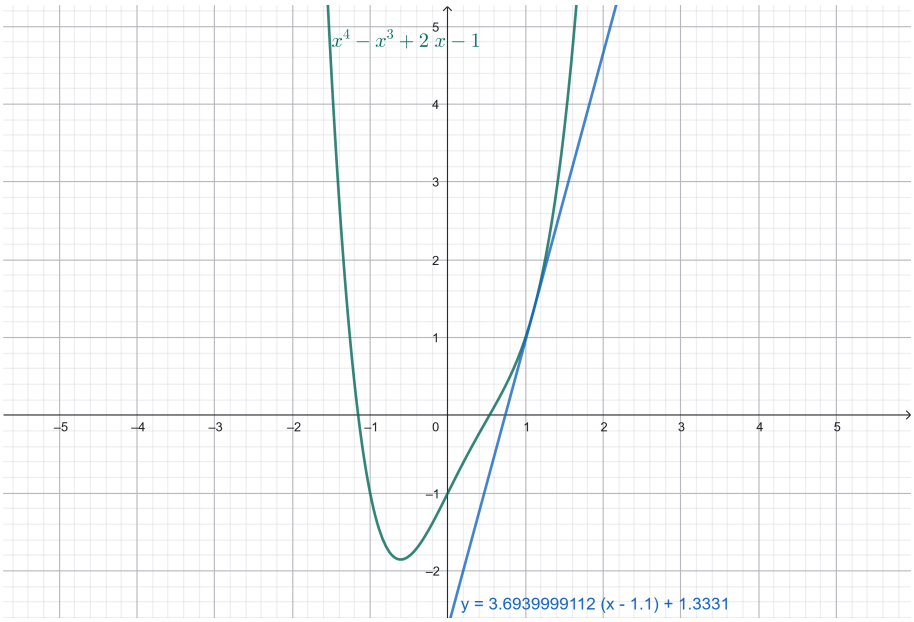
Therefore both lines; secant and tangent; are "overlapping parallel lines".

*** Computational Calculus of Derivative using Parallel Line Definition**

Next are shown the results to perform the method described early to define the derivative with a parallel line , using Open Office Calc. The results are graphed in Geogebra.

Open Office Calc Book to Calculate Derivative by Parallel Line

Geogebra online graph for $f(x) = x^4 - x^3 + 2x - 1$ and derivative on point (1.1 , 1.3331)



Derivative of $f(x) = x^4 - x^3 + 2x - 1$ on the point (1.1 ; 1.3331)

The function used was

$$f(x) = x^4 - x^3 + 2x - 1$$

with the coordinate ($x = 1.1$; $f(x) = 1.3331$) like point of derivation

with a value of $\delta = 0.000000001$

for 10 points before the coordinate of derivation; i.e.

$$(x - n * \delta, f(x - n * \delta))$$

and 10 points after; i.e.

$$(x + n * \delta, f(x + n * \delta))$$

with $n = \{1, ..., 10\}$

$m = \frac{y-1.3331}{x-1.1}$	$m_{\parallel} - m$
3.6939999284	-1.72306533485767E-008
3.6939999379	-2.6694688681772E-008
3.6939998857	2.54796259646639E-008
3.6939999455	-3.43217436693521E-008
3.6939998885	2.26485563636913E-008
3.693999884	2.71782676364296E-008
3.6939998942	0.000000017
3.6939997405	1.70678291500082E-007
3.6939998772	0.000000034
3.6939994671	4.44089173434747E-007
#DIV/0!	#DIV/0!
3.6940003553	-4.44089172990658E-007
3.6940003213	-4.1011638707289E-007
3.6940000366	-1.25381179216788E-007
3.6940001162	-2.05058181990125E-007
3.6940000616	-1.50457417191774E-007
3.6940000366	-1.25381179216788E-007
3.6940000724	-1.61204376514235E-007
3.6940001077	-1.96564978960367E-007
3.6940000859	-1.74724424262251E-007
3.6940000616	-1.50457417191774E-007

using the method described on the section of derivative with parallel line; obtaining a y-intercept of

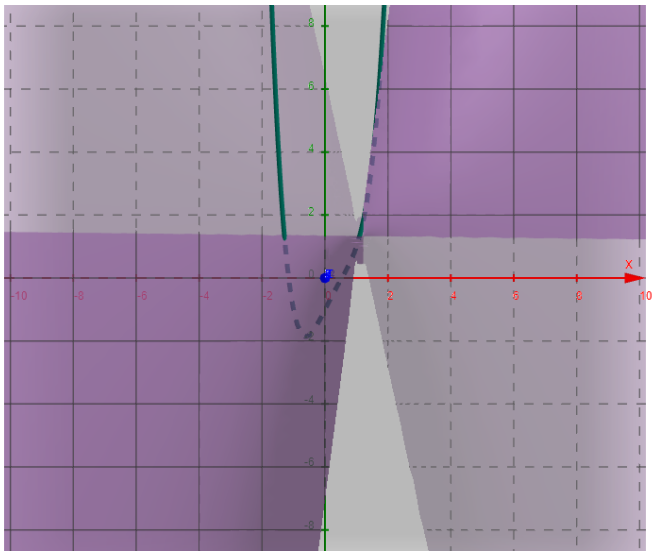
$$m_{\parallel} = 3.6939999112$$

Next, a table containing the difference between the y-intercept of each coordinate used to define the function $f(x) = x^4 - x^3 + 2x - 1$ is shown.

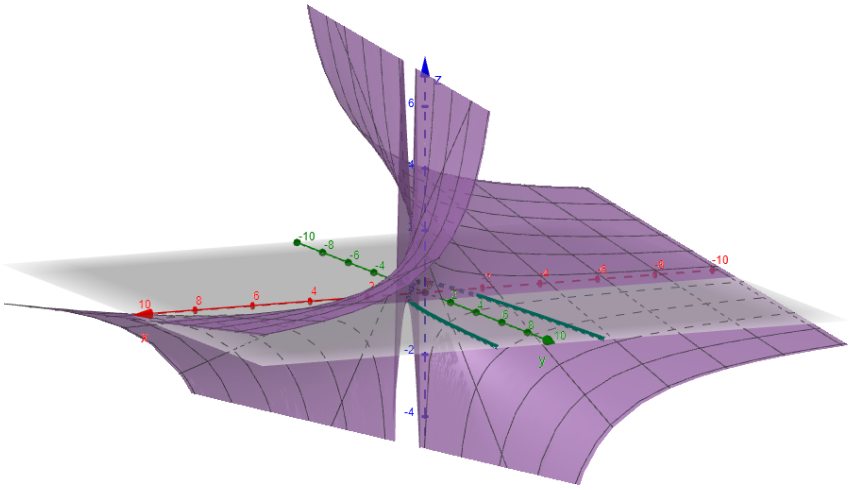
Only in the case of the same point of derivation the y-intercept is non defined; this means that this point is the only that is part of the straight line equation that defines the "tangent line".

In the other cases exist a difference between such points; i.e. any one of them are part of such line.

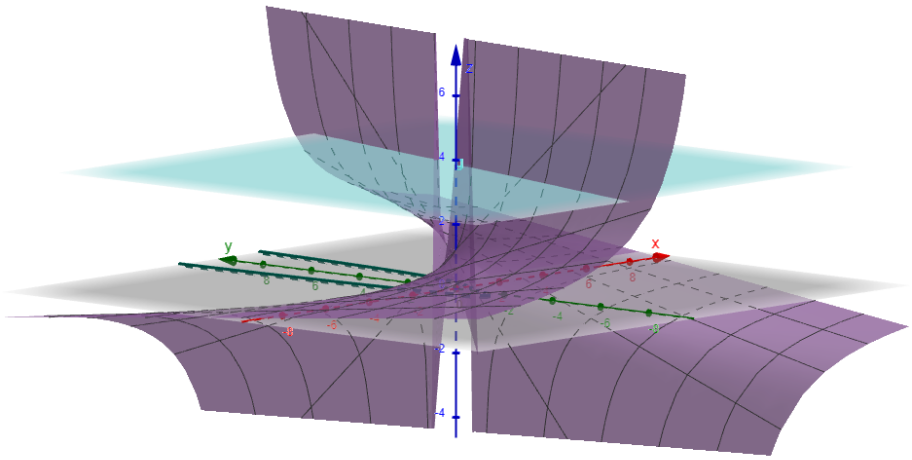
Graph in 3D at Geogebra



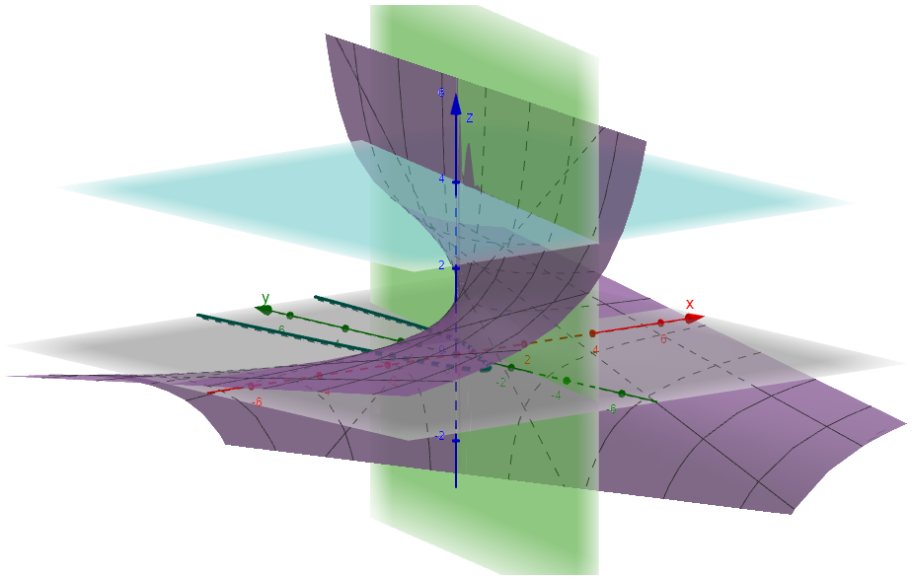
Graph 1 . $f(x) = x^4 - x^3 + 2x - 1$ and $z = \frac{y-1.3331}{x-1.1}$



Graph 2. $f(x) = x^4 - x^3 + 2x - 1$ and $z = \frac{y-1.3331}{x-1.1}$



Graph 3. Plane at $z = 3.6939999112$



Graph 4. Plane at $y = m_{\parallel}(x - 1.1) + 1.3331$

Can be see that on the plane XY a tangent line is shown on the graph 3D (Graph 1).

Such line can be defined at $z = 3.6939999112$ when such plane intersect the

function $z = \frac{y-1.3331}{x-1.1}$ (Graph 3).

Such line at the plane $z = 3.6939999112$ and the intersection of $z = \frac{y-1.3331}{x-1.1}$ in $\Re^3$ is part of next set (scheme 4)

$$y_{plane} = \{x \in \Re \quad | \quad m_{\parallel} = \frac{y - 1.3331}{x - 1.1} \quad ; \quad \forall z \in \Re^{+}\} \quad ;$$

Must be understood that the "parallel derivative" is a tangent line only when the difference between y-intercepts is zero (or a non defined value like is #DIV/0!); and that the accuracy of such method (defined with δ) is related with the number of points used to define the coordinates of the function. This means that the line is tangent if the data obtained in Open Office Calc are considered. [see early Open Office Calc book]

Note that graph on $\Re^2$ (scheme 1 and 4) and graph in $\Re^3$ are the "same".

* Close of ω

Lets consider the point $(x_1^*, f(x_1^*))$

where

$$f(x_1^*) = x_2^*$$

with

$$x_1^* \wedge x_2^* \in \bigwedge^2 (X)$$

the y-intercept of the tangent line at $(x_1^*, f(x_1^*))$ when is considered equal to the derivative of the function in such point is

$$m_{\parallel} = \frac{(f(x_1^*) + d_3 cos(\angle d_1 d_3) sin(\theta)) - (f(x_1^*) - d_2 cos(\angle d_1 d_2) sin(\theta))}{(x_1^* + d_3 cos(\angle d_1 d_3) cos(\theta)) - (x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta))}$$

when $\theta = 0$ (or $sin(0) = 0$)

hence

$m_{\parallel} = 0$

also when

$\angle d_1 d_3 = 0$ and $\angle d_1 d_2 = 0$

hence

$m_{\parallel} = 0$

i.e. the derivative of the function $f(x)$ at the point $(x_1^*, f(x_1^*))$ is equal to zero

Def (Hyp.) The exterior derivative of a differential form ω of degree k (also differential $k-$ form, or just $k-$ form for brevity here) is a the derivative of the exterior product $a \wedge b$ when $b = f(a)$ and a $2-$form ω of degree k.

Proof.

Remember that $\omega :=$ *form of degree* $k = 2$ is defined with

*) the smooth section

$$\sigma : CSB_{(h_1,h_2)} \to E$$

of the

*) *2 - exterior power* of the cotangent bundle

$$a \wedge b := \begin{cases} (a,b) \in X \times X \subseteq CSB_{(h_1,h_2)} & ; \\ ae_i + be_j = c \in R = T^2(X) & ; \\ c_1 - c_2 \in I \quad \text{such that} \quad c = xr(\text{rx}) \end{cases}$$

Hence the exterior derivative of the 2-form ω is

*) *2 - exterior power* of the cotangent bundle

$$d(a\wedge b) := \begin{cases} (a,b) \in X \times X \subseteq CSB_{(h_1,h_2)} & ; \\ b = f(a) & ; \\ m_{\parallel} = \frac{y_{\parallel_{\delta+}} - y_{\parallel_{\delta-}}}{x_{\parallel_{\delta+}} - x_{\parallel_{\delta-}}} & ; \\ y = m_{\parallel}(x-a) + b := \text{tangent line} \in \Re^2 & ; \\ z = \frac{y-f(a)}{x-a} \in \Re^3 & ; \\ z = m_{\parallel} \text{ line defined by interseccion of planes} & ; \\ \text{intersecion line is part of: } y = (x-a) + b := \text{tangent plane} \in \Re^3 & ; \\ ae_i + be_j = c \in R = T^2(X) & ; \\ c_1 - c_2 \in I \quad \text{such that} \quad c = xr(\text{or rx}) \end{cases}$$

with (a, b) on the smooth section

$$\sigma : CSB_{(h_1,h_2)} \to E$$

with

$$f(a) \in \Re^n$$

a function that is only dependent of "a"

However can have more variables (see section about "use of parallel derivative definition in functions of n variables")

Example 1.

*) Electrical resistance of a uniform conductor

In terms of resistivity such relation have the equation

$$R = \rho \frac{l}{a}$$

where

$a :=$ cross sectional area (for a round wire $a = \pi r^2$)

$l :=$ is the length of the conductor in SI units of meters

$\rho = i$ the resistivity in unit of ohms· meters

Like

$\frac{V}{I} = R$

If voltage (V) would like to be studied hence next equation

$V = I\rho\frac{l}{a}$

Could be explored

Lets consider that $V = g$ is function of 4 variables hence

$$g : (x, y, z, h) \to g(x, y, z, h);$$

$$g = x \cdot y \cdot \frac{z}{h}$$

Note that g could values can establish next cases

*) be dependent of the 4 variables at same time

*) be dependent of 3 of 4 variables at same time

*) be dependent of 2 of 4 variables at same time

*) be dependent of 1 of 4 variables

while the other could be considered like constants

In the two first cases graphics can no be elaborated

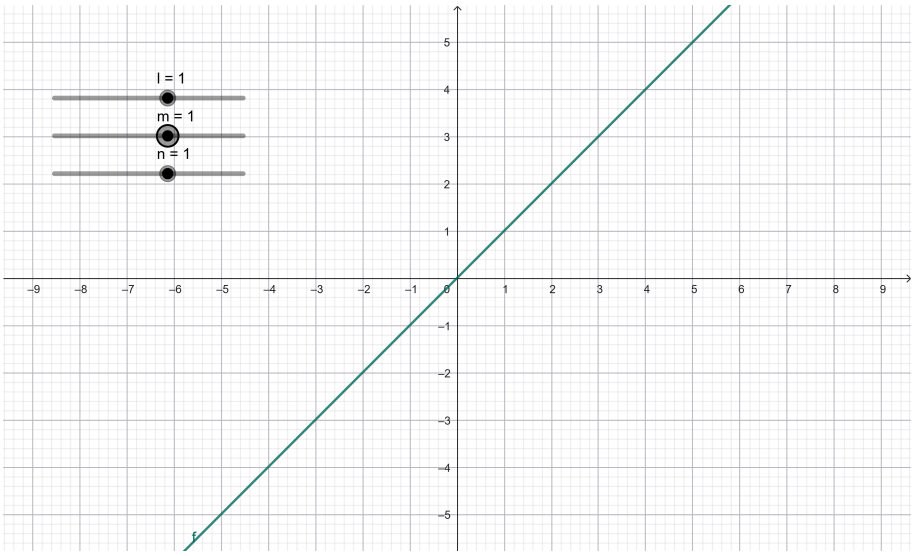
In the third case a graph in $\Re^2$ can be established

In the fourth case a graph in $\Re$ can be established

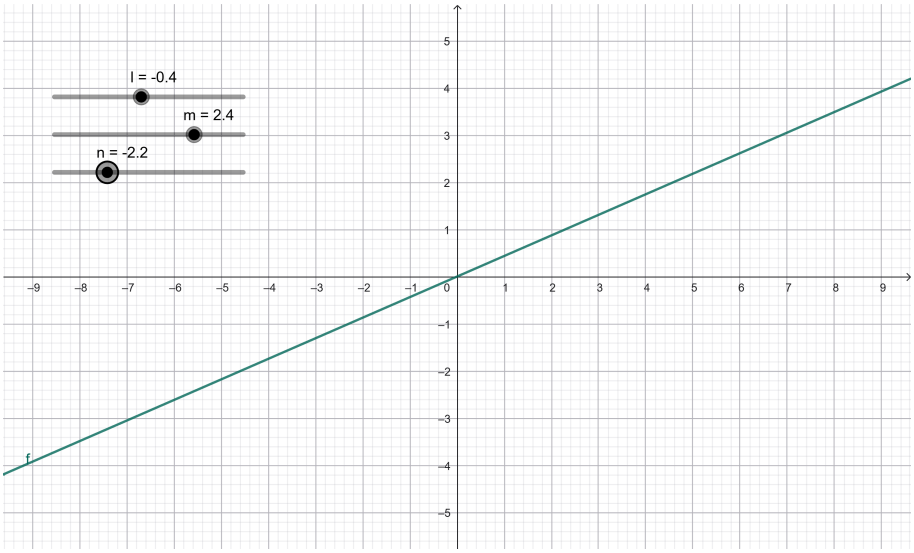
Note that this situation can be found if the values of such variables are defined like coordinates of an space or system of reference where each value establish a length to define a "common point" between all of them.

Note also that if the graphic is elaborated on a software like "Geogebra" other kind of graphs can be obtained when such variables are considered at same time

Graph of $g = x \cdot y \cdot \frac{z}{h} = x \cdot l \cdot \frac{m}{n}$ at Geogebra



Graph of g function



Graph of g function with values of l, m and n

When the values of l, m and n change the line defined by x (independent variable) rotates around the point $(0, 0)$.

This could give place to next hypothesis

Hyp. The image of $g(x, y, z, h)$ can be established considering the rotation of the line $y = ax$ with $y-$intercept $a = l\frac{m}{n}$.

Example 2.

The energy distribution of emission of a black hole is described by Planck´s law with a temperature T :

$$T = \frac{\hbar c^3}{8\pi G k_B M}$$

where

$\hbar :=$ Planck´s constant quotient by 2π

$c :=$ speed of light

$G :=$ gravitational constant

$k_B :=$ Boltzmann constant

$M :=$ mass of black hole

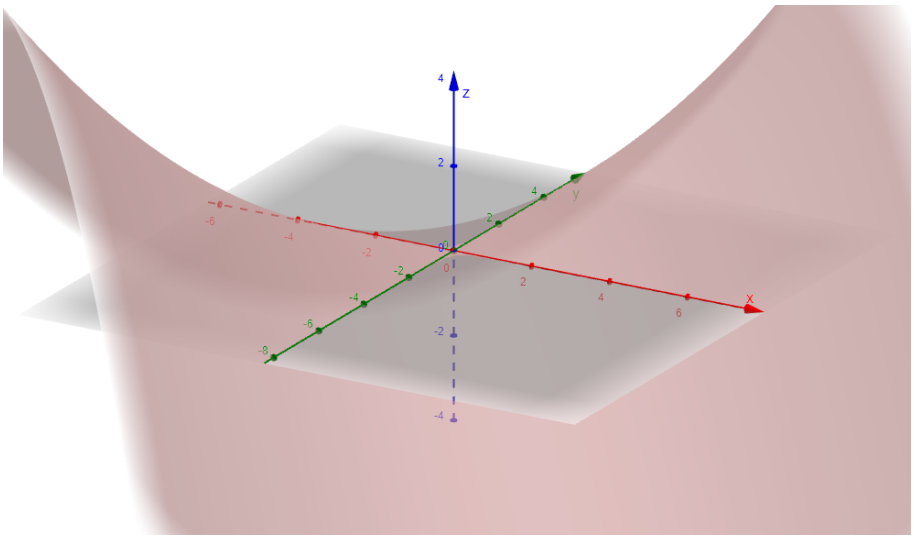
Overlooking that c, $\hbar$, k_B and G are constant, the equation of T can be expressed like

$$g : (x, y) \to g(x, y)$$

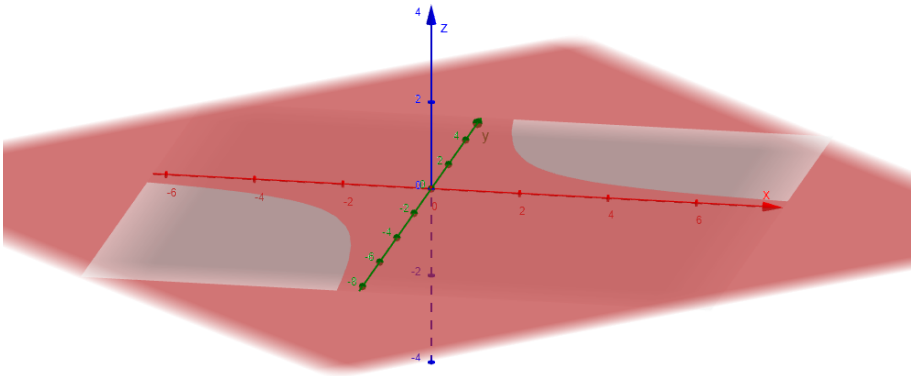
$$z = \frac{x \cdot y}{8\pi lmn}$$

with next graph

Graph of $z = \frac{x \cdot y}{8l \cdot n \cdot m}$ at Geogebra



Graph of z



Graph of z with l, m and n

i.e the image of the function z could be described through description of the "movement" of the plane shown on "graph of z with l, m, and n"

Def. In discrete mathematics, especially combinatorics and finite probability theory, n-tuples arise in the context of various counting problems and are treated more informally as ordered lists of length n.

Def. Properties of n-tuple. Thus a tuple has properties that distinguish it from a set:

*) A tuple may contain multiple instances of the same element, so tuple

$(1, 2, 2, 3) \neq (1, 2, 3)$; but set $\{1, 2, 2, 3\} = \{1, 2, 3\}$

*) Tuple elements are ordered: tuple

$(1, 2, 3) \neq (3, 2, 1)$, but set $\{1, 2, 3\} = \{3, 2, 1\}$.

*) A tuple has a finite number of elements, while a set or a multi-set may have an infinite number of elements.

$$-0-$$

If the dominium of the function is considered like a set that contains n-tuples then some problems about gratification could be understood.

To define the geometrical object that describe a function of n variables, a graphical representation of the derivatives through tangent lines at every stage of evaluation of the function, through elaboration of a pseudocode and its computational software will be introduced on next section.

Such results will be used to establish the meaning of the pullback function and cohomology classes

$$-0-$$

hence when

$$z = 0$$

therefore

$$d(x_1^* \wedge x_2^*) = 0$$

Hence the differential 2-form is closed

Hence the differential 2-form is symplectic

Hence $(E = \mathbb{C} = CSB_{(h_1,h_2)} \quad ; \quad \bigwedge^2(X))$ is a symplectic manifold.

$-o-$

5 Kähler Manifold is a Complex Manifold

*) Integrable Almost- Complex Structure

Def. Let M be a smooth manifold. An almost complex structure J on M is a linear complex structure (that is, a linear map which squares to -1) on each tangent space of the manifold, which varies smoothly on the manifold.

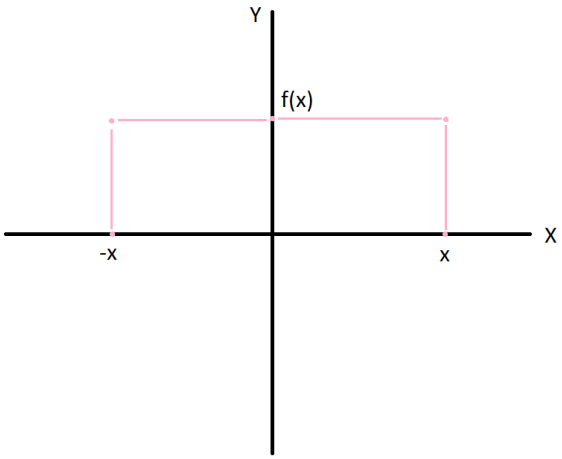
If M admits local holomorphic coordinates for J around every point then these patch together to form a holomorphic atlas for M giving it a complex structure, which moreover induces J.

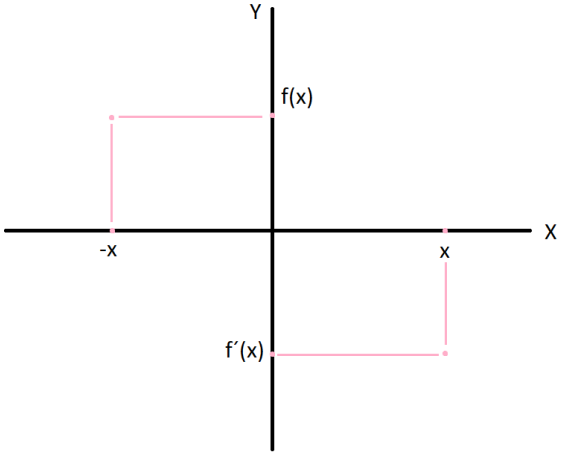
J is then said to be integrable.

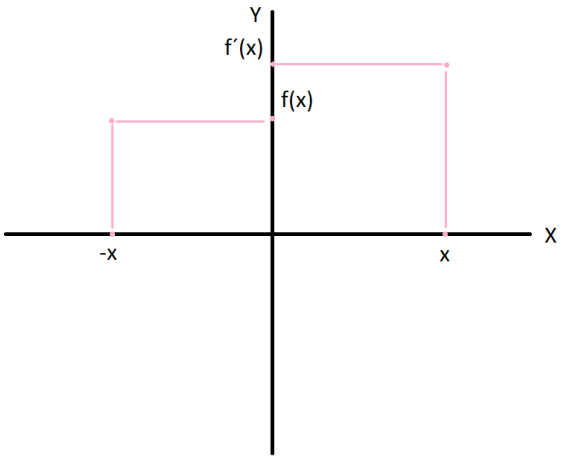
* Holomorphic Coordinates

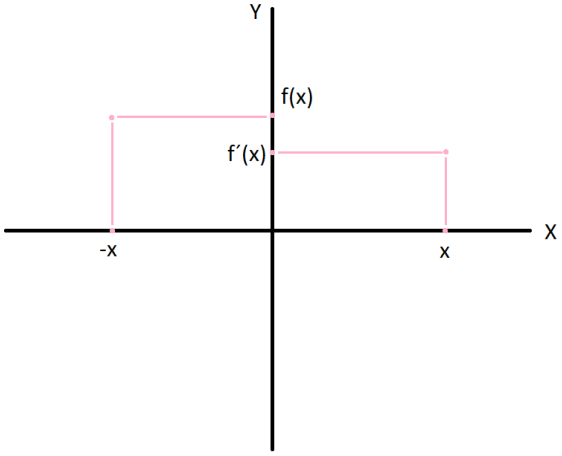
Proof.

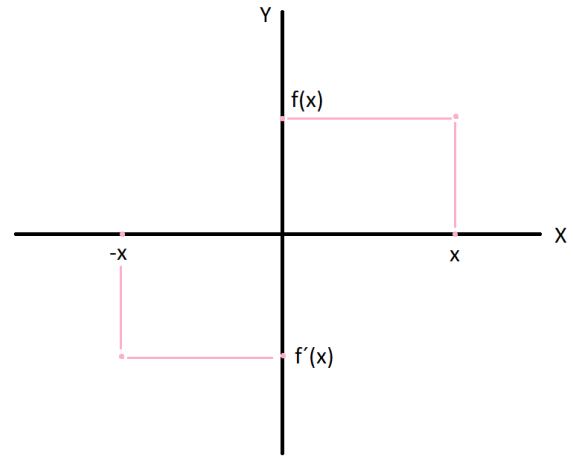
Lets consider next cases to compare two points on the Cartesian plane on $\Re^2$ with the properties of "abstract equality" $=_A$; i.e. lets consider next properties about coordinates on $\mathbb{C}$.

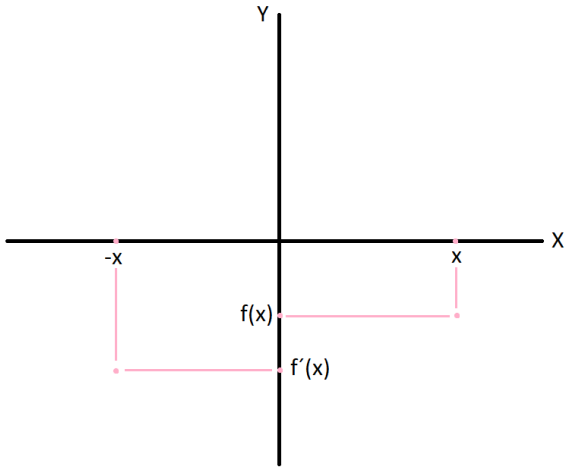

Y
f(x)
X
-x
x

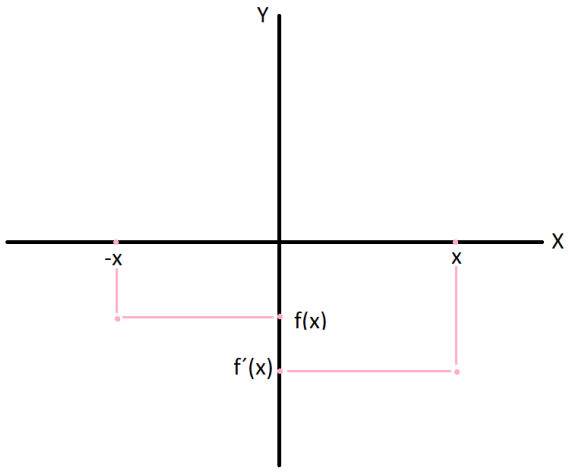

Y
f(x)
X
-x
x
f′(x)


Y
f′(x)
f(x)
X
-x
x


Y
f(x)
f′(x)
X
-x
x


Y
f(x)
X
-x
x
f´(x)


Y
X
-x
x
f(x)
f´(x)



1) $(-x, f(-x))$; $(x, f(x))$

$$-x =_A x \quad (-x < x) \quad ;$$

$$f(-x) = f(x)$$

2) $(-x, f(-x))$; $(x, -f(x))$

$$-x =_A x \quad (-x < x) \quad ;$$

$$f(-x) =_A -f(x) \quad (f(-x) > -f(x))$$

3) $(-x, f(-x))$; $(x, -f(x))$

$$-x =_A x \quad (-x < x) \quad ;$$

$$f(-x) > -f(x)$$

4) $(-x, f(-x))$; $(x, f(x))$

$$-x =_A x \quad (-x < x) \quad ;$$

$$f(-x) < f(x)$$

5) $(-x, f(-x))$; $(x, f(x))$

$$-x =_A x \quad (-x < x) \quad ;$$

$$f(-x) > f(x)$$

6) $(-x, -f(-x))$; $(x, f(x))$

$$-x =_A x \quad (-x < x)$$

$$-f(-x) =_A f(x) \quad (-f(-x) < f(x)) \quad ;$$

7) $(-x, -f(-x))$; $(x, f(x))$

$$-x =_A x \quad (-x < x) \quad ;$$

$$-f(-x) < f(x)$$

8) $(-x, -f(-x))$; $(x, -f(x))$

$$-x =_A x \quad (-x < x) \quad ;$$

$$-f(-x) < -f(x)$$

9) $(-x, -f(-x))$; $(x, -f(x))$

$$-x =_A x \quad (-x < x)$$

$$-f(-x) > -f(x)$$

Considering early cases, hence a vector on $\mathbb{C}$ can be defined having next properties:

*) Magnitude

Taken the derivative parallel definition for an holomorphic function given before, can be established that the size of the vector can be measured through the definition of distance.

Like derivative on the point $(x, f(x))$ is established through the secant line of the function $f(x)$ parallel to the line that pass through the point of derivation $(x, f(x))$ hence a distance can be established for such line through:

$d[(x - \delta, f(x - \delta)); (x + \delta, f(x + \delta))]$:= distance between the points of the secant line.

*) Direction

Like direction is defined with the y-intercept of the line hence; this property can be established through the angle of vector projection to the X axis.

Hence

θ := angle of projection on X

Like the position of the points on the Cartesian plane and the comparison between points considers the definition of the "abstract equality" hence a definition of coordinate on $\mathbb{C}$ can be established; considering that the value of the independent variable x of a function $f(x)$ can be compared through trichotomy and "abstract equality" hence the value of $f(x)$ is determined through this condition.

This means that both values of the entries on a vector are defined like complex numbers.

Like an holomorphic function is defined like

$$\Psi(x, f(x)) := g(Prod(x, f(x)))$$

Therefore the vector coordinate $(x, f(x))$ is an holomorphic coordinate.

*) Holomorphic atlas

Def. Let M be a topological space. A **holomorphic atlas** is a set of charts $\{(U_\alpha, \phi_\alpha)\}$ such that:

- Each $U_\alpha \subset M$ is open.
- Each $\phi_\alpha : U_\alpha \to \Re$ is a homeomorphism onto its image.
- For any overlapping charts $U_\alpha \cap U_\beta \neq \emptyset$, the **transition map**:

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is **holomorphic**.

Proof

See def.

* Has been proved early that U_α is open on $\mathbb{C}$

* Also has been proved that ϕ_α is an homeomorphism

and

* The transition map is an homeomorphism

Like

$(x, f(x)) \in U_\alpha$ is an holomorphic coordinate and

$(x, f(x)) \in U_\alpha \cap U_\beta$

and

$(x, f(x)) \in U_\alpha \cup U_\beta$

are holomorphic coordinates for every $U_\alpha \in \tau$

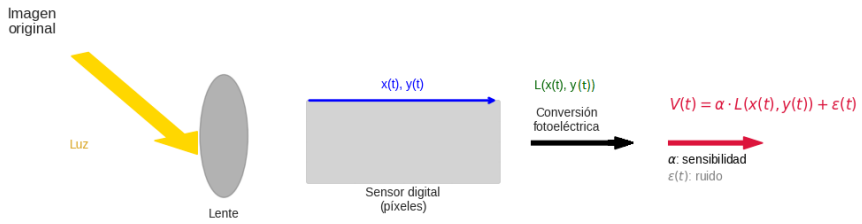
Hence the atlas is holomorphic

*) **Sense**

**) $M = \mathbb{C}$ is a foliation

- **Background:** -

Considers an image captured by a camera



Scheme of a camera capturing an image

Lets note that the light of the image pass through a lens and later is captured by a photodiode (digital sensor).

After this process a signal can be define to establish the voltage of each photodiode that produces a pixel. Such function can be defined like:

$$V(t) = \alpha \cdot L(x(t), y(t)) + \epsilon(t)$$

where

$V(t) :=$ Voltage of each photodiode

$L(x, (t), y(t)) :=$ Spatial Luminance Function

$\epsilon(t) :=$ Noise (error)

$\alpha :=$ Scale factor

Note that

$L(x(t), y(t))$ is function of $(x(t), y(t))$ a parametrization of the coordinates in terms of time

Also note that

$$L(x(t), y(t)) = h(x(t), y(t))$$

i.e. is an holomorphic function

Like $L(x(t), y(t))$ can be defined for each coordinate hence a matrix of luminance can be defined.

With M_{ij} such matrix hence

$$M_{ij} = \begin{pmatrix} L_{11} & \cdots L_{1n} \\ \vdots & \\ L_{n1} & \cdots L_{nn} \end{pmatrix}$$

And normalizing the value of each $L(x(t), y(t))$ with

$$G(x, y) = \frac{L(x(t), y(t)) - L_{min}}{L_{max} - L_{min}} \cdot 255$$

where each value of L_{ij} represent a value on "gray scale"; hence each term is such that

$$L(x(t)), y(t)) \in \{0, 1, 2, ..., 255\}$$

Lets suppose that on an "image" more than one value of L_{ij} will be equal hence

Taking

$$D = \{(i, j) | 1 \leq i \leq m, 1 \leq j \leq n\}$$

Like the domains of the matrix

The luminance function is

$$L : D \to \{0, 1, \cdots, 255\}$$

such that

$$L(i, j) = M_{ij}$$

Hence M_{ij} can be decompose by tone (on gray scale)

For each tone $k \in \{0, 1, \cdots, 255\}$, defines the set

$$\Omega_k = \{ \quad (i, j) \in D \quad | \quad L(i, j) = k \quad \}$$

Where Ω_k sets are disjoint and its union cover all the domain

$$\bigcup_{k=0}^{255} \Omega_k = D$$

and

$$\Omega_k \cap \Omega_{k'} = \emptyset, \quad k \neq k'$$

Note that also a decomposition by range of tones can be made

With

$$\{[a_r, b_r]\}_{r=1}^{R}$$

a set of disjoint ranges hence

$$\Delta_r = \{(i, j) \in D \quad | \quad L(i, j) \in [a_r, b_r]\}$$

defines such decomposition.

Note that

$[a_1, b_1] = [0, 85] \to$ dark tones

$[a_2, b_2] = [86, 170] \rightarrow$ medium tones

$[a_3, b_3] = [171, 255] \rightarrow$ light tones

Note that both decompositions defines a foliation

Now lets consider that on each plane a process of "detect silhouttes" is made; for example consider "Sobel algorithm"

Since the luminance matrix is establish

$$L : D \rightarrow R,$$

$$D = \{(i, j) | 1 \leq i \leq m, 1 \leq j \leq n\}$$

Taking the "Sobel´s masks" on a matrix of 3 x 3 for the same number of values of L_{ij}

$$G_X = \begin{bmatrix} -1 & 0 & +1 \\ -2 & 0 & +2 \\ -1 & 0 & +1 \end{bmatrix} \quad ;$$

$$G_Y = \begin{bmatrix} +1 & +2 & +1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix} \quad ;$$

For each point (i, j) the convolution is calculating like

$$S_X(i, j) = \sum_{k=-1}^{1} \sum_{l=-1}^{1} G_X(k+1, l+1) \cdot L_{i+k, k+l}$$

$$S_Y(i, j) = \sum_{k=-1}^{1} \sum_{l=-1}^{1} G_Y(k+1, l+1) \cdot L_{i+k, j+l}$$

also known as

* Horizontal Derivatives

* Vertical Derivatives

While the magnitude of the gradient is

$$G(i,j) = \sqrt{S_X(i,j)^2 + S_Y(i,j)^2}$$

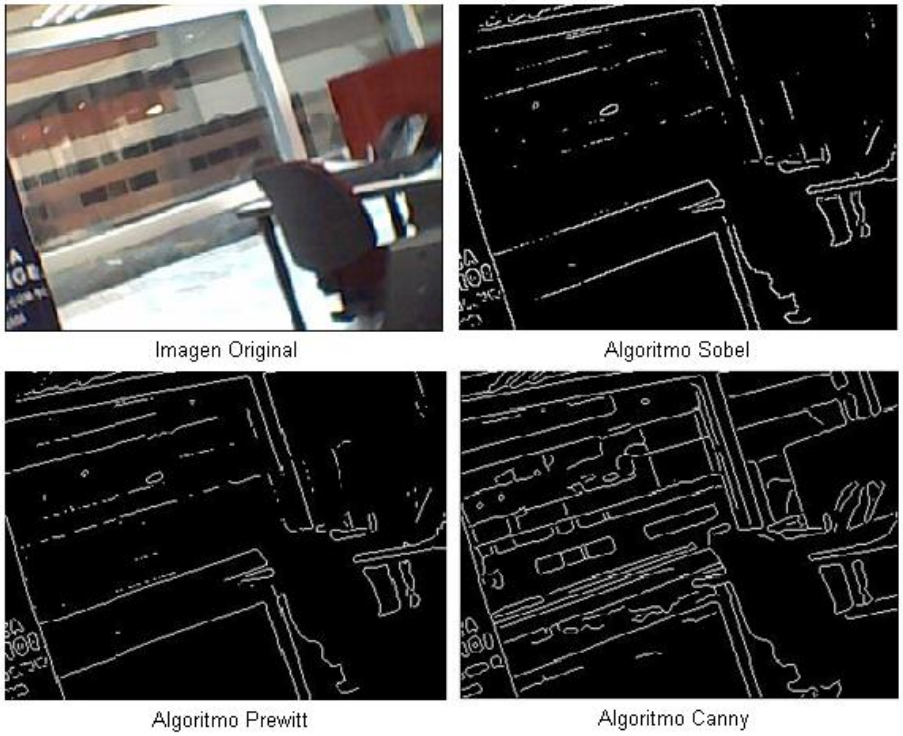
and the gradient direction is

$$\theta(i,j) = arctan(\frac{S_Y(i,j)}{S_X(i,j)})$$

If $G(i,j) \geq T$ hence (i,j) its considered part of the silhouette

Note that T could be chosen without any criteria

For example "the silhouette for $T = 100$"



Ref. Implementación de interfaz gráfica para comparación visual de métodos de segmentación y procesamiento de video, usando matlab - Scientific Figure on

ResearchGate. Available from: https://www.researchgate.net/figure/Figura-6-Deteccion-de-Bordes-usando-diferentes-algoritmos$_f ig4_277829411[accessed31Oct2025]$

Now lets consider that each silhouette can be define by pieces by means of straight lines.

Considering that each coordinate point $(x(t), y(t))$ have a derivative defined with

$y(t) = f(x(t))$:= the straight line equation;

$y(t) = m_{\|} \cdot x(t) + b$

and $m_{\|}$ defined by

$$1) \quad (x(t) - \delta, f(x(t) - \delta))$$

and

$$2) \quad (x(t) + \delta, f(x(t) + \delta))$$

note now that also the straight line equation defined with 1) and 2) describe such curve "silhouette".

Note that to define each straight line its length must be established through the interval of values of x that defines the distance between 1) and 2).

* Covector $q(t)$

Consider now the properties of the covector $q(t)$ for each coordinate $(x(t), y(t))$

Like

$$SB^* = \{q(t) \in Q(t) \quad | \quad R(\theta)\vec{v} = q(t) \cdot R(\theta)\vec{v}\}$$

with

$\vec{v} = (x, f(x)) \in \mathbb{C}$

$x \in \text{Dom}f$

$f(x) \in \mathrm{Im} f$

Lets consider that $q(t)$ is the "electric charge"

Like

$$q(t) = \int_0^t I(\tau) d\tau$$

where

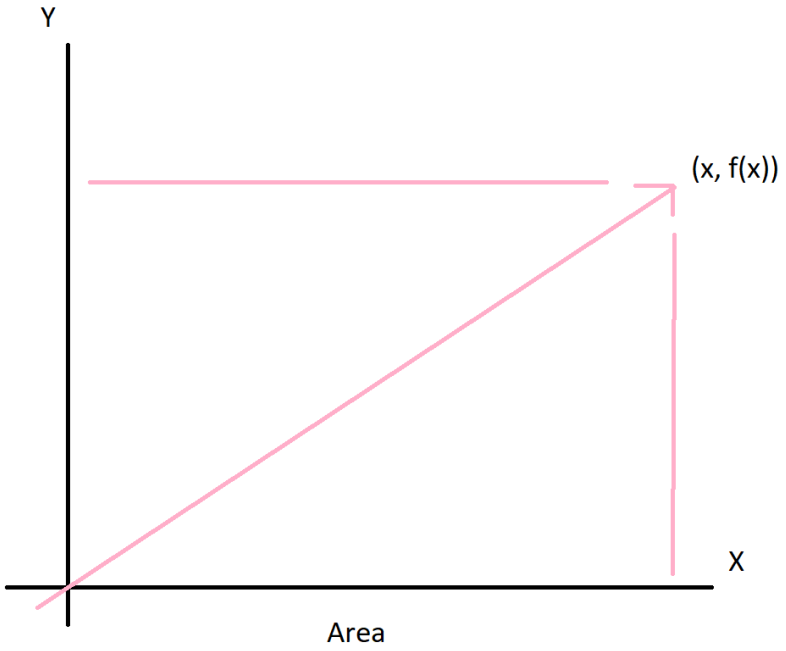
$I(\tau) :=$ electrical intensity

$t :=$ time

$\tau :=$ time of use of the current

* If each photodiode defines a pixel, hence each photodiode establishes a value of $I(\tau)$; i.e. the charge per pixel is equal to the last equation; and the value of each $q(t)$ can be related to its coordinate like a covector by the relation of $R(\theta)$ of the set SB^* of covectors; else this value can be related with the value of the area that defines a vector on the cartesian plane and also the area of each pixel by establishing the relation;

$\rho_{surface} = \frac{q(t)}{Area} :=$ surface density

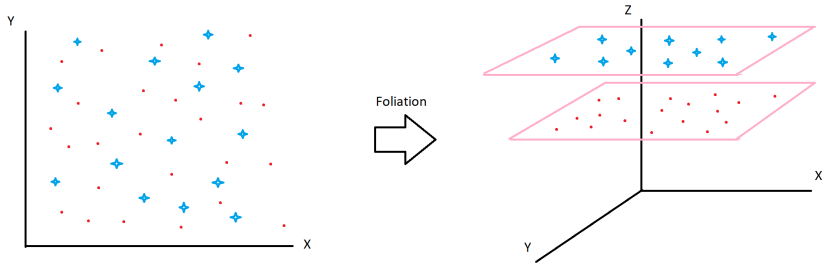


Pixel Area

* If the "scanning" of the image is made through "rasterization"; the value of the charge is established since the first pixel from "left to right" and since "up to down"; i.e. the value of the electric charge increases during the scan"; and the value of the charge in every pixel is equal to the integration of the electrical intensity through the time.

Definition of Foliation.

Def. A p-dimensional, class C^r foliation of an n-dimensional manifold M is a decomposition of M into a union of disjoint connected submanifolds $\{L_\alpha\}_{\alpha\in A}$, called the leaves of the foliation, with the following property: Every point in M has a neighborhood U and a system of local, class C^r coordinates $x = (x^1, \cdots, x^n) : U \to R^n$ such that for each leaf L_α, the components of $U \cap L_\alpha$ are described by the equations x^{p+1} =constant, , x^n =constant. A foliation is denoted by $\mathcal{F} = \{L_\alpha\}_{\alpha\in A}$.



Foliation Scheme

Lets take the equation of "Time-dependent scanned luminous acquisition model" and lets clear $L(x(t), y(t))$

Hence

$$L(x(t), y(t)) = \frac{V(t) - \epsilon(t)}{\alpha}$$

Remembering that voltage establish ref[1]

$$V = \frac{d(\bar{E}_1 - \bar{E}_2)}{q}$$

With

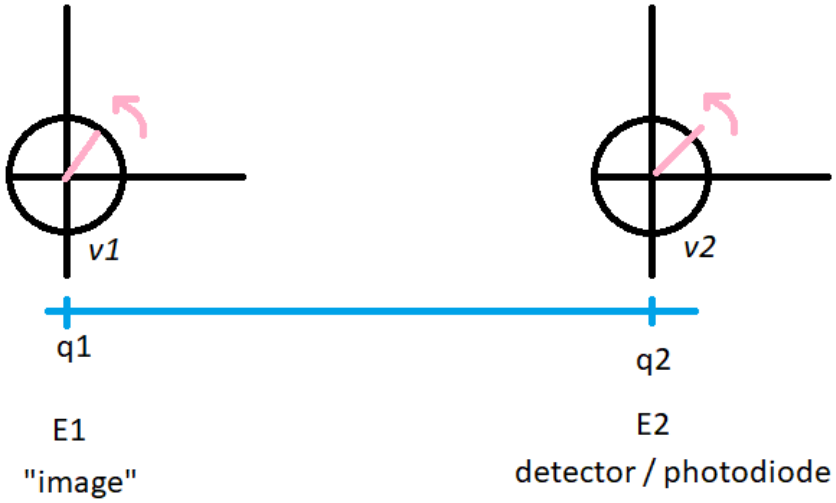
$$I = I_0(e^{\frac{qV}{nK_BT}} - 1)$$

the "Time-dependent scanned luminous acquisition model"

where

$q :=$ elemental charge

hence q of voltage relation is the elemental charge



This means that the charge per point of the image establish the voltage of detection through such relation i.e.

two points of an electric field can be defined

E_1 := point on the "image"

E_2 := point on the detector / photodiode

Like each $q(t)$ is related with a coordinate hence $L(x(t), y(t)) = h(x(t), y(t))$ can be defined like a function of two variables.

Like ref[1]

$$q(t) = A\frac{1}{f(\frac{dx_1}{dt}, \frac{dx_2}{dt})}$$

where

$A = (\frac{n}{q_{e^-}}) \cdot (\frac{h\nu}{m_{\bar{e}}})$

ν := wave frequency

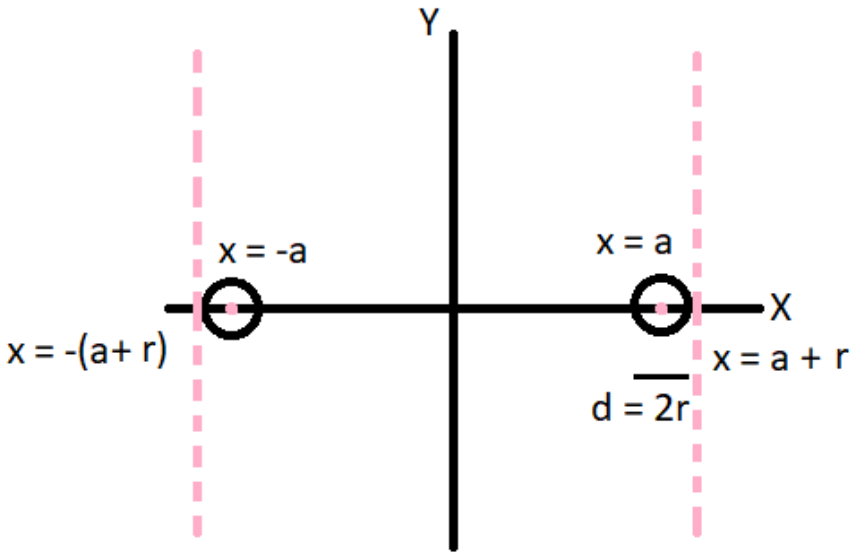
$\frac{dx_i}{dt}$:= velocity

when limited of dx and dt is considered; such quotient describe light velocity

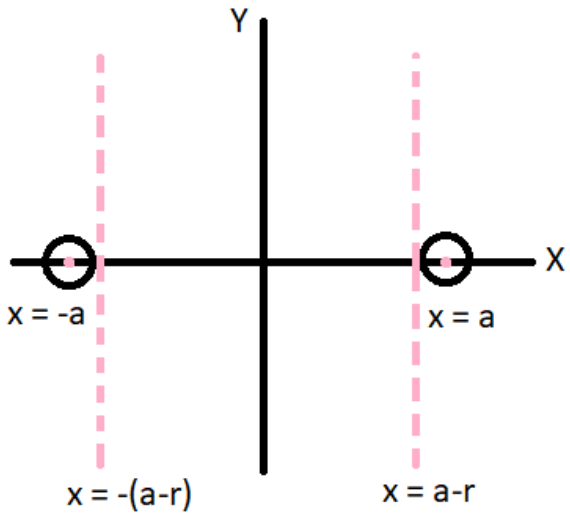
Now consider the definition of cover given at the beginning of this work with

$$(x-a)^2 + (y-b)^2 = r^2$$

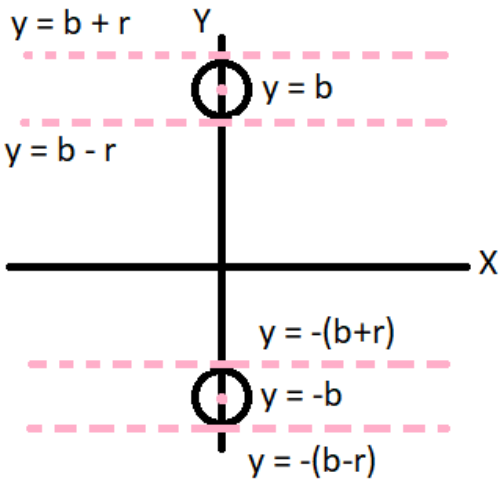
the equation of the circle lets consider the open sets on $a = 0$ and $b = 0$ *2 *



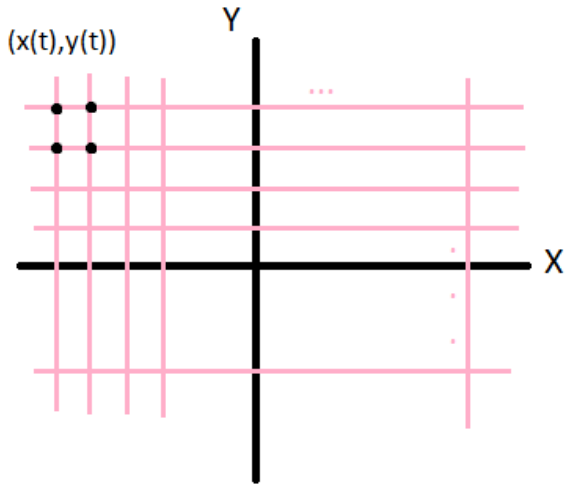
cover



cover



cover



grid

Now consider the interval of the vertical lines (for the line of $x = a + r$)

$$-na - r \leq x_1 \leq na + r \quad ;$$

$$-\infty < x_2 < \infty$$

besides consider the interval of horizontal lines

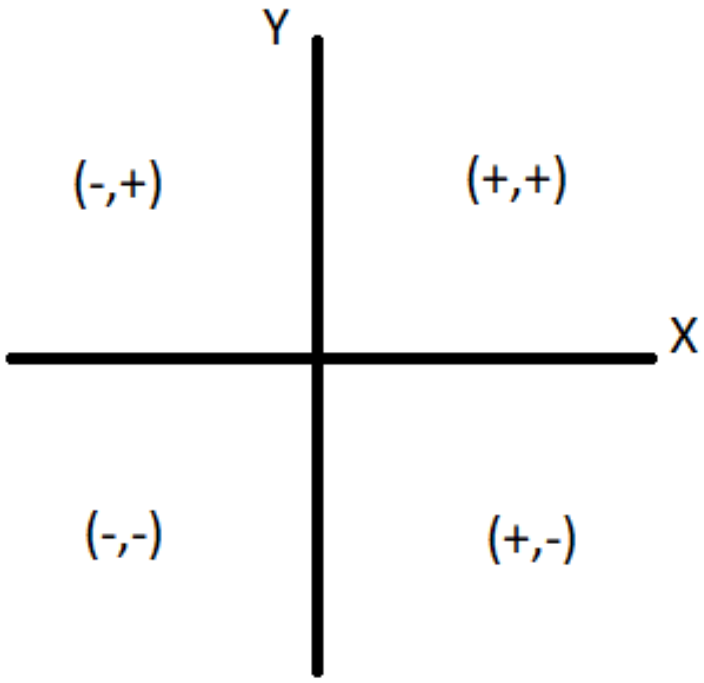
$$-\infty < x_1 < \infty \quad ;$$

$$-nb - r \leq x_2 \leq nb + r$$

With

$$n \in \mathbb{N}$$

And the coordinates by section



$$y_{\text{top},+} = nb + r \quad ;$$

$$y_{\text{bottom},+} = (n-1)b + r \quad ;$$

$$x_{\text{outside},+} = na + r \quad ;$$

$$x_{\text{inside},+} = (n-1)a + r \quad ;$$

$$y_{\text{bottom},-} = -nb - r \quad ;$$

$$y_{\text{top},-} = -(n-1)b - r$$

$$x_{\text{outside},-} = -nb - r \quad ;$$

$$x_{\text{inside},-} = -(n-1)b - r$$

Now consider that the center of each pixel is defined by the coordinate (x, y) with

$$x = na \quad \text{and} \quad y = nb$$

With area equal to

$$Area = |(y_{top} - y_{bottom})| \cdot |(x_{outside} - x_{inside})|$$

(or $Area = |(y_{bottom} - y_{top})| \cdot |(x_{outside} - x_{inside})|$)

Like each (x, y) have a value of $q(t)$ to be its covector; considering that the image has been decomposed by range of colors (or tones in "gray scale")

Hence the open set defined for each $q(t)_i \in [a_i, b_i]$ is

$$B_r(q_i) = \{q_x \in Q(t) \quad | \quad d(q_i, q_x) < r\}$$

Like

$$*) \quad q(t)_i \in B_r(q_i) \cap B_r(q_i') \in \tau_i$$

$$**) \quad q(t)_i \in B_r(q_i) \subset \tau_i \quad , \bigcup_{B_r(q_i') \in B_r(q_i)} B_r(q') \in \tau_i$$

Hence each τ_i is a topological space

where the τ_i set are disjoint between them

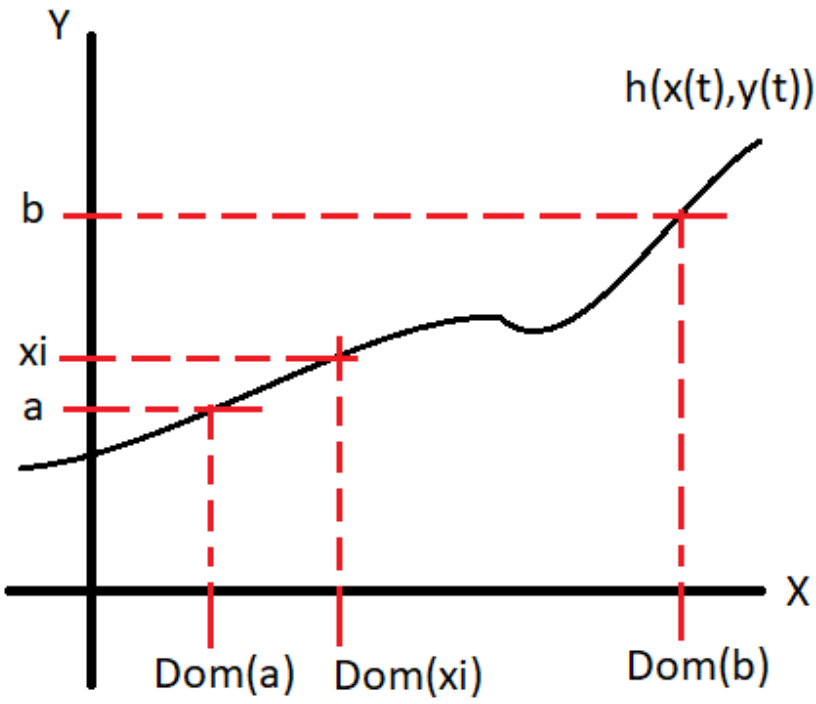
*)

Now considers the function $h(x(t), y(t)) \in [a, b]$; where

$[a, b] :=$ Range of the function

and

$$h(x(t), y(t)) := g(Prod(x(t), y(t)) = g(x(t) \cdot y(t)) = g(x(t)) * g(y(t))$$



Graph of the function h(x(t),y(t))

Now considers the interval $[Dom(a), Dom(b)]$ where the normalization of the value and escalation is

$$G(x(t), y(t)) = \frac{h(x(t), y(t)) - h_{min}}{h_{max} - h_{min}} \cdot f_v$$

where f_v establish the total of values of a distribution of the random variable $h(x(t), y(t))$

Now considers the value per unit of G in terms of degrees

$$\theta = \frac{G_{max} - G_{min}}{360}$$

with

$$G_{min} \quad \text{at} \quad \theta = 0$$

and

$$G_{max} \quad \text{at} \quad 360$$

i.e.

$$G_{max} - G_{min} = s$$

where

$s :=$ arc length

is the length of the perimeter of a circle

and

$$\theta = \frac{(G_i - G_j)}{G_{max} - G_{min}} \cdot (360)$$

is the value of angle between each pair of points of the random distribution for a circle of radius 1.

Like each $L(x(t), y(t)) := h(x(t), y(t))$ defines a value of θ hence each coordinate $(x(t), y(t))$ have a value of θ (each pixel) i.e. have a direction

*)

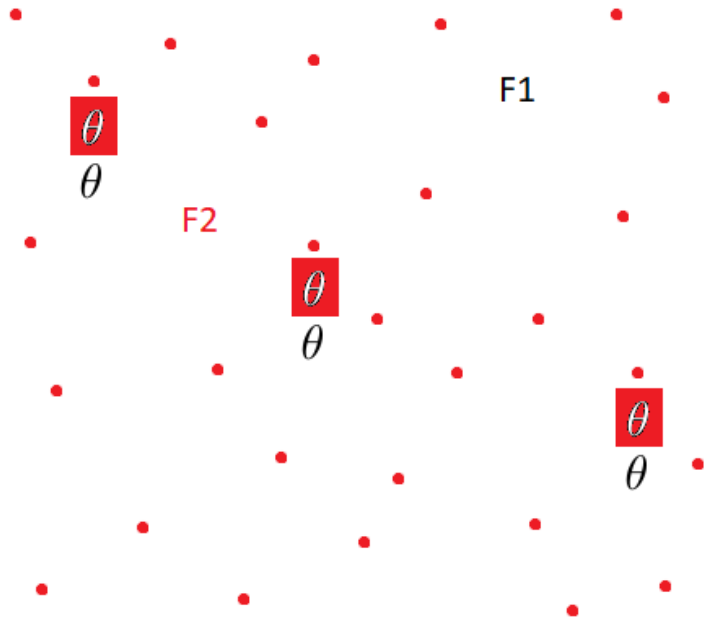
Now considers the "inverse-square law"

Def. The inverse-square law generally applies when some force, energy, or other conserved quantity is evenly radiated outward from a point source in three-dimensional space. Since the surface area of a sphere (which is $4\pi r^2$) is proportional to the square of the radius, as the emitted radiation gets farther from the source, it is spread out over an area that is increasing in proportion to the square of the distance from the source. Hence, the intensity of radiation passing through any unit area (directly facing the point source) is inversely proportional to the square of the distance from the point source.

$$\text{intensity} \propto \frac{1}{\text{distance}^2}$$

Proof.

Now take into consideration $\vec{F}_1$ and $\vec{F}_2$ two vector fields; where $\vec{F}_1$ is the vector field defined with the direction; sense and magnitude for each $(x(t), y(t))$ on the dominium of $h(x(t), y(t))$ and F_2 like next:



Vector fields; $\vec{F}_2$ with θ white in red rectangle; $\vec{F}_1$ with θ in black

Now considers that $\vec{F}_2$ field is defined by the vector function

$$\vec{F_2} = \frac{B}{r^2} \quad \vec{r}$$

where

B is function of $q(t)$

with

$$q(t) = A \cdot \frac{1}{f(\frac{dx_1}{dt}, \frac{dx_2}{dt})}$$

and A is function of ν

and considers that

$$||\vec{F_2}|| = |\frac{A}{r^2}| \cdot ||\vec{r}|| \quad ;$$

with

$||\vec{r}||$:= vector norm

on $\Re^2$

$$|F_x| = |\vec{F_2}| \cdot cos(\theta') \quad ;$$

$$|F_y| = |\vec{F_2}| \cdot sin(\theta')$$

such that on $(x(t), y(t))$ the intensity h(x(t),y(t)) is the sum of the "intensity" of the two fields; i.e.

$$h(x(t), y(t)) = ||\vec{F_1}|| + ||\vec{F_2}||$$

With the first value of $h(x(t), y(t))$ defining the rotation matrix with θ

hence the relation

$$R(\theta)\vec{v} \quad = \quad q(t) \quad R(\theta')\vec{v}$$

is established with

θ' defined by $h(x(t), y(t)) = ||\vec{F_1}|| + ||\vec{F_2}||$

and the value of $q(t)$ that maintains the equality

Like the norm is equal to the euclidean distance hence

$$V = \alpha \cdot h(x(t), y(t)) + \epsilon(t)$$

where

$$\epsilon(t) = V(t) - \hat{V}(t)$$

and

$$\hat{V} = \alpha \cdot (h(x(t), y(t)))$$

hence

$$d(\text{Field}_1, \text{Field}_2) = \alpha \cdot h(x(t), y(t)) + \epsilon(t)$$

with

$d(\text{Field}_1, \text{Field}_2)$:= distance between two points of a vector field

*)

Now lets establish the relation between both vectors and the derivative

with

$$\gamma = Prod(x(t), y(t))$$

and the intensity function

$$h(x(t), y(t)) = g(Prod(x(t), y(t)) = g(\gamma)$$

Like $h(x(t), y(t))$ is a random variable X; a PDF (probability density function if X is continuous) or a PMF (probability mass function if X is discrete) is necessary to establish its definition.

Properties of a Probability Density Function and Probability Mass Function

For

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

The next are properties

*) $f(x) \geq 0$ for all x

*) $\int_{-\infty}^{\infty} f(x) \quad dx = 1$

*) $P[a \leq X \leq b] = \int_a^b \quad f(x) \quad dx = F(b) - F(a)$

If the variable is discrete with $PMF := f(x)$ such that

*) $P(X = x_i) = f(x_i)$

*) $\sum_i f(x_i) = 1$

$-0-$

Hence like

$$h(x(t), y(t)) = P[a \leq X \leq b]$$

hence

$$\int_a^b \quad f(x) \quad dx = h(x(t), y(t)) \in [a, b]$$

remember that

$$m_{\|} = \frac{(f(x_1^*) + d_3 cos(\angle d_1 d_3) sin(\theta)) - (f(x_1^*) - d_2 cos(\angle d_1 d_2) sin(\theta))}{(x_1^* + d_3 cos(\angle d_1 d_3) cos(\theta)) - (x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta))}$$

with

$$G(x(t), y(t)) = \frac{h(x(t), y(t)) - h_{min}}{h_{max} - h_{min}} \cdot f_v$$

$$= \frac{P[a \leq X \leq b] - P[a \leq X \leq b]_{min}}{P[a \leq X \leq b]_{max} - P[a \leq X \leq b]_{min}} \cdot f_v$$

where

$[a, b]$ establish an interval on the domain of $h(x(t), y(t))$

$[a, b]_{min}$ the minimum interval on the domain of $h(x(t), y(t))$

$[a, b]_{max}$ the maximum interval on the domain of $h(x(t), y(t))$

and with

$$\theta = \frac{G_{max} - G_{min}}{360} \quad ;$$

substituting on the equation of $m_{\|}$; hence the relation between derivative and the random variable $h(x(t), y(t))$ its established with the value of $P[a \leq X \leq b]$.

Such probability will be considered like the "sense" of the vector.

*) $M = \mathbb{C}$ is a Pfaffian system

Def. A Pfaffian system is specified by 1-forms, defining a set of 1-forms on a smooth manifold.

Given a collection of differential 1-forms $\alpha_i, i = 1, 2, ..., k$ on an $n-$dimensional manifold M, an integral manifold is an immersed (not necessarily embedded) submanifold whose tangent space at every point $p_T \in N$ is annhilated by (the pullback of) each α_i.

A maximal integral manifold is an immersed (not necessarily embedded) submanifold

$$i : N \subset M$$

such that the kernel of the restriction map on forms

$$i^* : \Omega^1_{p_T}(M) \to \Omega^1_{p_T}(N)$$

is spanned by the α_i at every point p_T of N. If in addition the α_i are linearly independent, then N is $(n-k)$- dimensional.

A Pfaffian system is said to be completely integrable if M admits a foliation by maximal integral manifolds. (Note that the foliation need not be regular, i.e. the leaves of the foliation might not be embedded submanifolds.)

An integrability condition is a condition on the α_i to guarantee that there will be integral submanifolds of sufficiently high dimension.

Proof.

Remember that in ref[1] was proved that $Q(t)$ is a topological space and a topological manifold.

Now lets take $M = \mathbb{C}$ and next definition

Def. An **immersed submanifold** (now a vector space) of a (vector space) differential manifold M is a submanifold (sub vector space) that is establish like the image of an smooth immersion:

$$f : N \to M$$

where

* N is other differential manifold (now other vector space)

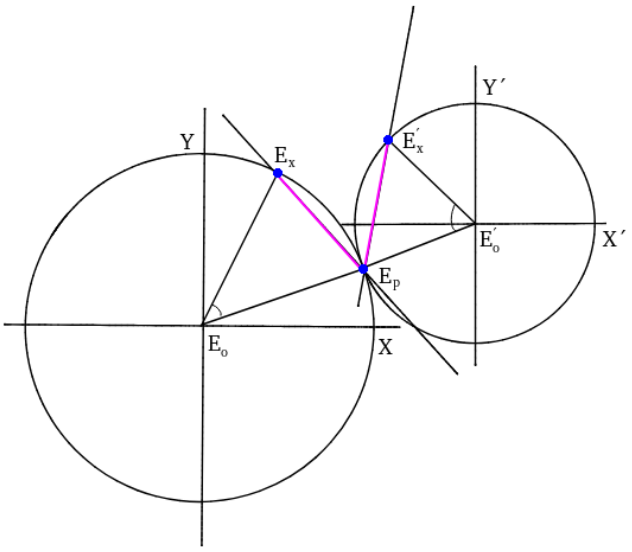
* The image $f(N)$ its considered a submanifold (sub vector field) of M, however self-intersections (auto-intersections) can be established or not being embedded.

With $\mathbb{C}$ a differential manifold; lets take ref[1]

$$S_pM = \{(x_1', x_2') \in \mathbb{C} \quad | \quad d[(x_1, x_2), (x_1', x_2')] = 2r^2(1-\gamma) \neq 0; \angle(x_1, x_2), (a, b), (x_1', x_2') \neq 0\}$$

with $r = 10^i$ or 10^{-i} and p the center of such open sets

$$B_r(p) = B(p; r) = \{x \in X : d(x, p) < r\}$$



Secant space ref[1] where $p := E_0$

Now lets establish the set $T_{p_T}M$

$T_{p_T}M = \{ \quad p_T := (x_1^*, f(x_1^*)) \in \mathbb{C} \quad |$

$$p := (x_1, x_2) \in S_pM \quad ;$$

$$[x_1 = x_1^* - \delta] < x_1^* < x_1^* + \delta \quad ;$$

$$x_2 = f(x_1^* - \delta) < f(x_1^*) < f(x_1^* + \delta)$$

(or

$$x_1^* - \delta < x_1^* < (x_1 = x_1^* + \delta) \quad ;$$

$$f(x_1^* - \delta) < f(x_1^*) < [x_2 = f(x_1^* + \delta)] \qquad) \quad ;$$

$$m_{\|} = \frac{(f(x_1^*) + d_3 cos(\angle d_1 d_3) sin(\theta)) - (f(x_1^*) - d_2 cos(\angle d_1 d_2) sin(\theta))}{(x_1^* + d_3 cos(\angle d_1 d_3) cos(\theta)) - (x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta))} \quad ;$$

$$y = m_{||}(x - x_1^*) + f(x_1^*) \quad ;$$

$$(x, y) \in \mathbb{C}$$

$$\}$$

Now lets establish the "annihilation" of $T_{p_T}M$ by the pullback of each α_i

Def. Pullback (Before) Let $f : N \to M$ be a smooth mapping between differentiable manifolds (now vector fields), and let ω be a k-differential form on M (now the covector set of a differential manifold).

The pullback of ω by f, denoted $f^*\omega$, is a k-form on M defined by:

$$(f^*\omega)_p(v_1, ...v_k) = \omega_{f(p)}(df_p(v_1), ..., df_p(v_k))$$

with

* df_p is the derivative of f at the point p, which transforms tangent vectors of N into tangent vectors of M.

* The pullback preserves the shape type (a k-form remains a k-form) but moves it to the domain of f.

Def. (Now). Let $f : N \to M$ be a smooth mapping between differentiable manifolds (now vector fields), and let ω be a k-differential form on M (now the covector set of a differential manifold).

The pullback of ω by f, denoted $f^*\omega$, is a k-form on M defined by:

$$(f^*\omega)_p(v_1, ...v_k) \in N$$

* The pullback preserves the shape type (a k-form remains a k-form) but moves it to the domain of f.

Proof.

Like $\mathbb{C}$ is foliated by $q(t) \in Q(t)$

Hence each leaf is a subspace of the space $\mathbb{C}$

Now consider next definition

Def. If $N \subset M$ is a subspace, its annihilator is the set of covectors $\alpha \in N^*$ such that:

$$\alpha(n) = 0 \quad ; \quad \forall n \in N$$

this set is

$$N^0 = \{\alpha \in N^* \quad | \quad \alpha(n) = 0 \quad ; \forall n \in N\}$$

besides next definition

Def. Let be $T : N \to M$ between vector spaces

The **kernel** of T is the set of all vectors of N that maps to the vector zero of M:

$$ker(T) = \{n \in N \quad | \quad T(n) = 0\}$$

Proof.

In ref[1] was proved that $q(t)$ is a linear transformation

Taking

$T := q(t)$

Like $q(t)$ is also a covector

Hence

$$\alpha = T = q(t)$$

Like $T_{p_T}N \subset \mathbb{C}$

Hence the set of $(x, y) \in T_{p_T}N \subset \mathbb{C}$ with $q(t) = 0 \in T^*_{p_T}N$

establish the kernel

$$ker(q(t)) = \{(x, y) \in T_{p_T}N \quad | \quad q(t)(x, y) = \vec{0}\}$$

and the "annihilator"

$$T_{p_T}N^0 = \{ \quad q(t) \in T^*_{p_T}N \quad | \quad q(t)(x, y) = \vec{0} \quad ; \quad \forall (x, y) \in T_{p_T}N \quad \}$$

Hence the annihilator of $T_{p_T}N$ and the kernel of $q(t)$ are the same

Like

$$(x, y) \in \mathbb{C}$$

a function h with domain in $\mathbb{C}$ is such that

$$h(x, y) = g(Prod(x, y)) \in \Re$$

hence

$$h(T_{p_T}N) \not\subset \mathbb{C}$$

hence the immersion can not be established

instead

Def. (Before) Let N be a differential manifold and $T_{p_T}N$ its tangent space on the point p_T

f is an immersion if its differential $df_{p_T} : T_{p_T}N \to T_{f(p_T)}M$ is injective in each point $p_T \in N$

Def. (Now) Let M be a vector space and $T_{p_T}M$ its tangent space on the point p_T

f is an immersion if its pullback $f^*_{p_T}\omega : T_{p_T}N \to T_{f(p_T)}M$ is co-linear in each point $p_T \in N$ to other point of $p_c \in M$

Proof.

Like each point of $\mathbb{C}$ can be colinear by $q(t)$ ref[1]

hence

$$p_T = q(t) \cdot p_c$$

If

$$f : N \to M$$

with $N, M \subset \mathbb{C}$

and

$(x, y) \in Dom(f)$ within $q(t) \in T^*_{p_T}M$ establish last condition about colinearity

Hence such function is an immersion

Note that the space of tangent points establish the slope $m_{||}$ and hence the derivative of a function f on such point $(x, f(x))$ [instead to be immersed].

Also note that there no 1 - form because

$$(a, b) \in X \times X \subseteq CSB_{(h_1,h_2)}$$

and ω- form only can be defined on $CSB_{(h_1,h_2)}$

With

$T_{p_T}N$ the tangent space of $p_T := (x, y) \in Dom(h)$

Hence

$$N = Dom(h)$$

Hence $Dom(h) \subset M$

Hence $N \subset M$ is integral

* $N \subset M$ is maximal

Def. Maximal Integral Manifold. If N is integral and $N \subset N'$ with N', then $N = N'$

Proof.

Like $q(t) \in T^*_{p_T}N$ defines the subspace N

If $q(t)_1$ defines N_1

and

$q(t)_2$ defines N_2

Where

$q(t)_1 \neq q(t)_2$

Hence

$N \neq N'$

Therefore if

$N \subset N'$

Hence $N = N'$

Hence N is integrable maximal

* Pfaffian System

Proof.

Like $T_{p_T}N^*$ the set of smooth manifolds of dimension 1 defines the integrable maximal set N

Hence the Pfaffian system is defined with $T_{p_T}N^* \subseteq Q(t)$

Where $Q(t)$ is a smooth manifold ref[1]

* Pfaffian System is Completely integrable

Proof.

Like each $N \subset M$ is maximal integrable hence

With N a leaf

Hence M admits a foliation by integrable maximal subspaces

Hence the Pfaffian system is completely integrable

-o-

Def. Embedded. A subset $N \subset M$; is an embedded manifold if:

1. N has structure of differential manifold

2. If $i : N \hookrightarrow M$ is a smooth immersion (its derivative is injective on each point) and a homeomorphism between N and its image on M then i is an embedding.

-o-

$\mathbb{C}$ is integrable

Now lets define integration and its relation with derivative

*) Integration $\int_a^b P(x)dx = (b-a)\big[f(a) + \frac{f(b)-f(a)}{2}\big]$

Def. Trapezoidal Rule

Lets consider next polynomial defining the equation of a line joining the points $(a, f(a))$ and $(b, f(b))$

$$P_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

$$\int_a^b P(x)dx = \int_a^b f(a)dx + \int_a^b \frac{f(b) - f(a)}{b - a}(x - a)dx$$

with the first term constant on x:

$$\int_a^b f(a)dx = f(a)(b - a)$$

now lets note

$$m = \frac{f(b) - f(a)}{b - a}$$

$$\int_a^b m(x - a)dx = m\int_a^b (x - a)dx = m\Big[\frac{(x - a)^2}{2}\Big]_a^b = m \cdot \frac{(b - a)^2}{2}$$

Hence

$$\int_a^b \frac{f(b) - f(a)}{b - a}(x - a)dx = \frac{f(b) - f(a)}{b - a} \cdot (\frac{(b - a)^2}{2}) = \frac{(f(b) - f(a))(b - a)}{2}$$

Adding the results hence the integral of the straight line interpolator over [a,b] is

$$\int_a^b P(x)dx = f(a)(b - a) + \frac{(f(b) - f(a))(b - a)}{2} = (b - a)\big[f(a) + \frac{f(b) - f(a)}{2}\big]$$

When [a,b] is divided by n subintervals of same length

each sub interval have width

$$h = \frac{b-a}{n}$$

the points of separation are

$$x_0 = a, \quad x_1 = a + h, x_2 = a + 2h, ..., x_n = b$$

On each subinterval $[x_i, x_{i+1}]$ the trapezoidal rule gives

$$I = \int_{x_i}^{x_i+1} f(x)dx = \frac{h}{2}\big[f(x_i) + f(x_{i+1})\big]$$

Adding over all sub intervals

$$I = \sum_{i=0}^{n-1} \frac{h}{2}\big[f(x_i) + f(x_{i+1})\big]$$

Hence

$$I = \frac{h}{2}(f(x_0) + f(x_1) + f(x_2) + ... + f(x_{n-1}) + f(x_n))$$

Hence

$$I = \frac{h}{2}\big[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\big]$$

Proof.

Consider the definition given early to define the exterior derivative in terms of the pair $(x_1^*, f(x_1^*))$

with

$$L_{tan} := y = (x - x_1^*) \cdot m_{||} + f(x_1^*)$$

where

$m_{||}$:= slope of tangent line

the derivative is

$$f' = \frac{dy}{dx} = m_{||}$$

Now using the equation of the straight line that pass through two points [see scheme of tangent line and $f(x_1, x_2) := g(Prod(x_1, x_2))$]

the to points are

$(x + \delta, f(x + \delta))$ and $(x - \delta, f(x - \delta))$

hence the slope

$$m_{||} = \frac{f(x + \delta) - f(x - \delta)}{(x + \delta) - (x - \delta)}$$

using the point $(x - \delta, f(x - \delta))$

hence

$$y - f(x - \delta) = m_{||} \cdot (x - (x - \delta))$$

$$L_{sec} := y = m_{||} \cdot (x - (x - \delta)) + f(x - \delta)$$

Remember

Def. A function of a real variable $f(x)$ is differentiable at a point a of its domain, if its domain contains an open interval containing a, and the limit

$$L = lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

exists.

This means that,

$(\forall \epsilon > 0); (\exists \delta > 0)$ s.t. $(\forall h$ s.t. $|h| < \delta)$ and $(h \neq 0)$ then $f(a+h)$ is defined, and

$$\left| L - \frac{f(a+h) - f(a)}{h} \right| < \epsilon$$

If the function$f$ is differentiable at a, that is if the limit L exists, then this limit is called the derivative of f at a

$$-o-$$

Now lets take $(x - \delta, f(x - \delta))$ and $(x + \delta, f(x + \delta)) \in CSB_{(h_1,h_2)}$

such that

$$(x - \delta, f(x - \delta)) \in C_1'$$

and

$$(x + \delta, f(x + \delta)) \in C_2'$$

with

$$x - \delta < x + \delta \quad ;$$

$$x + \delta = x - \delta + h \quad ;$$

and

$$h = 2 \cdot \delta$$

Taking into consideration two preceding equations

$$y = P_1(x - \delta) \quad ;$$

and the definition of exterior derivative

$$m_{||} = \psi'(x_1^*, f(x_1^*)) = \frac{y - f(x_1^*)}{x - x_1^*} = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(x + \delta) - f(x - \delta)}{(x + \delta) - (x - \delta)} = \frac{f(x + h) - f(x)}{h}$$

If

$$h = [(x + \delta) - (x - \delta)] \to 0$$

hence

$$m_{||} = lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

therefore

$$L = m_{||} = \psi'(x_1^*, f(x_1^*)) \quad ;$$

Like ψ is differentiable hence L exists

Besides like $L = m_{||} \quad ;$

L is the exterior derivative $d(x_1^* \wedge f(x_1^*))$ of $(x_1^*, f(x_1^*))$

Hence

$$y = (x - (x - \delta)) \cdot L + f(x - \delta)$$

Now lets take the function $\psi(x_1^*, f(x_1^*))$ defined like before, such that

$$P_1(x - \delta) := y = f(x - \delta) + \frac{f(x + \delta) - f(x - \delta)}{(x + \delta) - (x - \delta)}(x - (x - \delta))$$

now consider $[(x-\delta),(x+\delta)] \in \Re$ an interval

for example with

$$x-\delta < x+\delta \quad ;$$

$$f(x-\delta) < f(x+\delta) \quad ;$$

Taking

* $(x-\delta)-(x+\delta)$ like width

and

$f(x-\delta)$ like the height of a rectangle

besides

** $(x+\delta)-(x-\delta)$ like width

and

$f(x+\delta)-f(x-\delta)$ like height of a triangle
Hence with

A_r denoting the area of the rectangle and A_t the area of the triangle, if follows that

$$A_r = ((x+\delta)-(x-\delta)) \cdot f(x+\delta) \quad ;$$

$$A_t = \frac{1}{2}((x+\delta)-(x-\delta)) \cdot (f(x+\delta)-f(x-\delta))$$

Now adding both areas

$$A_r + A_t = ((x+\delta)-(x-\delta)) \cdot f(x-\delta) + \frac{1}{2}((x+\delta)-(x-\delta)) \cdot (f(x+\delta)-f(x-\delta))$$

$$= ((x+\delta)-(x-\delta)) \cdot (f(x-\delta) + \frac{1}{2}f(x+\delta) - \frac{1}{2}f(x-\delta))$$

$$= ((x+\delta)-(x-\delta)) \cdot (\frac{1}{2}f(x-\delta) + \frac{1}{2}f(x+\delta))$$

$$= ((x+\delta)-(x-\delta)) \cdot \frac{1}{2}(f(x-\delta) + f(x+\delta))$$

Now let $\{a_k\}$ be a partition of $[(x-\delta),(x+\delta)]$ such that

$$(x-\delta) = a_0 < a_1 < \cdots < a_{n-1} < a_n = (x+\delta)$$

and $h_i = \Delta a_k$ be the length of the subinterval; that is $h_i = \Delta a_k = a_k - a_{k-1}$ hence

$$P_i(a) = f(a_i) + \frac{f(a_i - f(a_{i-1})}{a_i - a_{i-1}}(a - a_{i-1})$$

Hence

$$\int_{a_{i-1}}^{a_i} P_i(a)da = \frac{h}{2}[f(a_{i-1}) + f(a_i)]$$

Adding each term by interval

$$\sum_{k=1}^{N} \frac{f(a_{k-1}) + f(a_k)}{2} \cdot \Delta a_k = \sum_{k=1}^{N}(A_{r_k} + A_{t_k})$$

with

$$h = \Delta x = \frac{(x+\delta)-(x-\delta)}{n} = a_k - a_{k-1}$$

Hence $h = \frac{a_k - a_{k-1}}{2}$ is the midpoint of each subinterval

If $f(x)$ is linear hence

$$f(\frac{a_{k-1}+a_k}{2}) = \frac{f(a_k)+f(a_{k-1})}{2}$$

therefore

$$\sum_{k=1}^{N} \frac{f(a_k)+f(a_{k-1})}{2}\Delta a_k = \sum_{k=1}^{N} f(\frac{a_k+a_{k-1}}{2})\Delta a_k$$

with

$h_i = \frac{a_k+a_{k-1}}{2} \in [a_k, a_{k-1}]$

hence

h_i establish the midpoint rule of Riemann sum

hence

$\sum_{k=1}^{N} f(\zeta_i)\Delta a_k$ is a Riemann sum

hence $f(x)$ is integrable when Δa_k tends to zero if $f(x)$ is continuous; linear and is bounded on the interval of integration.

* Kähler manifold

Def. Since Kähler manifolds are equipped with several compatibles structures, they can be described from different points of view:

Symplectic viewpoint

A Kähler manifold is a symplectic manifold (X, ω) equipped with an integrable almost-complex structure J that is compatible (see eq 1) with the symplectic form ω, this means that the bilinear form

$$g(u, v) = \omega(u, Jv)$$

on the tangent space of X at each point is symmetric and positive definite and hence a Riemannian metric on X.

* Riemannian metric

Def. A Riemannian metric g on M assigns to each p a positive-definite symmetric bilinear form (i.e. an inner product)

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

in a smooth way (see the section on regularity below). A Riemannian metric is a special case of a metric tensor. A smooth manifold M endowed with a Riemannian metric g is a Riemannian manifold, denoted (M,g). A Riemannian metric is not to be confused with the distance function of a metric space, which is also called a metric.

Proof.

Lets take exterior product definition for $(x_i, f(x_i)) \in T_{p_T}M; \quad i = [I, II]$

$$a_i \wedge b_i := \begin{cases} i \in [I, II] & ; \\ a_i = x_i - \delta \quad [\text{ or } \quad a_i = x_i + \delta] & ; \\ b_i = f(x_i - \delta) \quad [\text{ or } \quad b_i = f(x_i + \delta)] & ; \\ (a_i, b_i) \in X \times X \subseteq CSB_{(h_1,h_2)} & ; \\ a_i e_i + b_i e_j = c \in R = T^2(X) & ; \\ c_1 - c_2 \in I \quad \text{such that} \quad c = xr(\text{rx}) \end{cases}$$

Now lets consider

$$detA =< (x_I, f(x_I)), (x_{II}, f(x_{II})) >= (x_I, f(x_I)) \otimes (x_{II}, f(x_{II})) = x_I f(x_{II}) - x_{II} f(x_I)$$

also remember that

$$J := e_{\Re}$$

i.e. $e_{(\Re,\bullet)}$ is the multiplicative identity on the reals; where $(\Re, \bullet)$ is an etiquette to define the subindex.

With the 2-form ω defined like the exterior product of the smooth section on the cosecant bundle $CSB_{(h_1,h_2)}$.

By integrable condition $x_i = q(t)(x_i - \delta)$ [or $x_i + \delta$]

2-form ω is defined on $T_{p_T}M$

Hence with $(x_I, f(x_I)), (x_{II}, f(x_{II})) \in T_{p_T}M$ by $CSB_{(h_1,h_2)}$

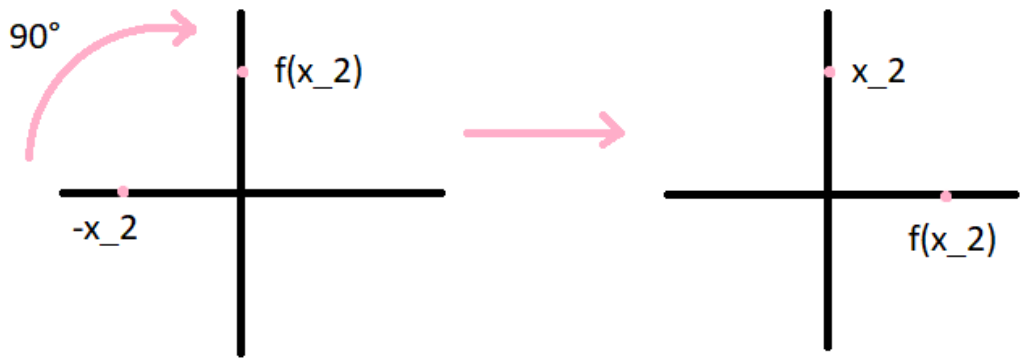
$$\omega((x_I, f(x_I)), J(x_{II}, f(x_{II}))) = \omega((x_I, f(x_{II})), (x_{II}, f(x_I I)) :=< (x_I, f(x_I)), (x_{II}, f(x_{II})) >$$

Now lets prove that the skew symmetric pairing is an inner product

Consider for next proof the pair (a, b) where "a" is an independent variable and b is dependent of a hence $b = f(a)$

+ Inner product

First lets note next diagram



Rotation of vector

Now consider

$$< (x_I, f(x_I)), (-x_{II}, f(x_{II})) > \quad = \quad x_I f(x_{II}) + f(x_I)x_{II}$$

$$= \quad (x_I, f(x_I)) \cdot (-x_{II}cos\theta + f(x_{II})sin\theta, (-1)(-x_{II}sin\theta + f(x_{II})cos\theta))$$

$$= (x_I, f(x_I)) \cdot R(\theta_{right})(f(x_{II}), x_{II})$$

where

$\cdot :=$ "inner cartesian product"

$\theta = 90$

and the subindex right represent a clockwise rotation with R the rotation matrix

$$R = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}$$

early definition about the skew-symmetric pairing and inner product is equal to

$$< (x_I, f(x_I)), (-x_{II}, f(x_{II})) > \quad = \quad (x_I, f(x_I)) \cdot (f(x_{II}), x_{II})$$

Now lets prove that the skew symmetric pairing is a norm

+ Norm

With $U = (x_I, f(-x_I))$ lets consider next equalities

$$||U||_p^2 = U \cdot U = (x_I, f(-x_I)) \cdot (x_I, f(-x_I))$$

$$= U \cdot R(\theta)V = (x_I, f(-x_I)) \cdot R(90)(-f(-x_I), x_I)$$

$$= (x_I, f(-x_I)) \cdot (-f(-x_I)cos\theta + x_I sin\theta, (-1)(-f(-x_I)sin\theta + x_I cos\theta)$$

$$= x_I^2 + f(-x_I)^2 = x_I x_I + f(-x_I)f(-x_I) = x_I x_I - (-f(x_I))f(-x_I)$$

$$= x_I x_I - f(-x_I)f(-x_I) =< (x_I, f(-x_I)), (f(-x_I), x_I) >$$

Hence

$$||U||_p = \sqrt{< (x_I, f(-x_I)), (f(-x_I), x_I) >}$$

Now consider next definition

Def. Let V be a vector space over a body $K = \Re$ or $\mathbb{C}$

$$|| \cdot || : V \to \Re \quad ;$$

$$x \mapsto ||x||$$

is a norm if verify

1) Non negativity

$$||x|| \geq 0 \quad ; \forall x \in V$$

Proof.
Lets name the square product of U like $||U||_p = ||g_p(U)||$

$$||g_p(x_I, f(-x_I))|| \quad = \quad \sqrt{< (x_I, f(-x_I)), (f(-x_I), x_I) >}$$

$$= \sqrt{x_I x_I - f(-x_I)f(-x_I)} > 0 \quad (\text{ or } = 0 \quad \text{if} \quad x_I^2 = f(-x_I)^2)$$

2) Homogeneity

$$||kx|| = |k| \cdot ||x||; \quad \forall x \in V; \quad \forall k \in K$$

Proof.

$$||k \cdot g_p(x_I, f(-x_I))|| \quad = \quad \sqrt{k \quad \cdot < (x_I, f(-x_I)), (f(-x_I), x_I) >}$$

$$= \sqrt{k \cdot (x_I x_I - f(-x_I) f(-x_I))} = \sqrt{k} \cdot \sqrt{x_I x_I) - f(-x_I) f(-x_I)}$$

$$= |k| \cdot \sqrt{< (x_I, f(-x_I)), (f(-x_I), x_I) >}$$

$$= |k| \cdot ||g_p(x_I, f(-x_I))||$$

3) Triangle inequality

$$||x + y|| \leq ||x|| + ||y||; \quad \forall x, y \in V$$

Proof.

$$||g_p(x_I, f(-x_I)) + g_p(x_{II}, f(-x_{II}))||^2$$

$$=< (x_I, f(-x_I)), (f(-x_I), x_I) > + < (x_{II}, f(-x_{II})), (f(-x_{II}), x_{II}) >$$

$$= x_I x_I - f(-x_I) f(-x_I) + x_{II} x_{II} - f(-x_I) f(-x_{II})$$

with $f(x)$ a linear function

$$= x_I x_I + f(x_I) f(-x_I) + x_{II} x_{II} + f(x_I I) f(-x_I I)$$

with $f(x) = f(-x)$; i.e. a pair function

$$= x_I x_I + f(x_I) f(x_I) + x_{II} x_{II} + f(x_{II}) f(x_{II})$$

$$= (x_I, f(x_I)) \cdot (x_I, f(x_I)) + (x_{II}, f(x_{II})) \cdot (x_{II}, f(x_{II}))$$

$$= ||g_p(x_I, f(x_I))||^2 + ||g_p(x_{II}, f(x_{II}))||^2$$

Now consider

$$(||g_p(x_I, f(x_I))|| + ||g_p(x_{II}, f(x_{II}))||)^2$$

$$= ||g_p(x_I, f(x_I))||^2 + ||g_p(x_{II}, f(x_{II}))||^2 + 2||g_p(x_I, f(x_I))|| \cdot ||g_p(x_{II}, f(x_{II})||$$

Hence adding the term to the first set of equation

$$||g_p(x_I, f(x_I)) + g_p(x_{II}, f(x_{II}))||^2 < ||g_p(x_{II}, f(x_{II}))||^2 + 2||g_p(x_I, f(x_I))|| \cdot ||g_p(x_{II}, f(x_{II})||$$

Simplifying

$$||g_p(x_I, f(x_I)) + g_p(x_{II}, f(x_{II}))||^2 < (||g_p(x_I, f(x_I))|| + ||g_p(x_{II}, f(x_{II}))||)^2$$

Obtaining square root

$$||g_p(x_I, f(x_I)) + g_p(x_{II}, f(x_I I))|| < ||g_p(x_I, f(x_I))|| + ||g_p(x_{II}, f(x_{II}))||$$

Hence, norm fulfills triangle inequality.

*Every norm establish a distance function

Proof.

* The distance function is asymmetric

Def. $d(x, y) := \|x - y\|$

-) Definite Positive; $d(x,y) \geq 0; \forall x \in V$

Proof.

$$||g_p(x_I, f(-x_I)) - g_p(x_{II}, f(-x_{II}))||$$

$$= \sqrt{< (x_I, f(-x_I)), (f(-x_I), x_I) > - < (x_{II}, f(-x_{II})), (f(-x_{II}), x_{II}) >} \quad > 0$$

if

$$x_I^2 + f(-x_{II})^2 > x_{II}^2 + f(-x_I)$$

else if

-) Identity of the indiscernible; $d(x,y) = 0$ iff $x = y$

$$x_{II} = x_I$$

Hence

$$||g_p(x_I, f(-x_I)) - g_p(x_{II}, f(-x_{II}))|| = \sqrt{x_I x_I - f(-x_I)f(-x_I) - x_I x_I + f(-x_I)f(-x_I)} = 0$$

-) Triangle inequality; $d(x,z) \leq d(x,y) + d(y,z)$

$$d((x_I, f(-x_I)), (x_{III}, f(-x_{III})))^2 \quad = \quad < (x_I, f(-x_I)), (f(-x_I), x_I) > - < (x_{III}, f(-x_{III})), (x_{III}, f(-x_{III}))$$

$$= ||(x_I, f(-x_I)||^2 - ||(x_{III}, f(-x_{III})||^2$$

Hence

$$d((x_I, f(-x_I)), (x_{III}, f(-x_{III})))^2 < (x_I, f(-x_I)), (f(-x_I), x_I) > - < (x_{II}, f(-x_{II})), (x_{II}, f(-x_{II})) >$$

$$+ < (x_{II}, f(-x_{II})), (x_{II}, f(-x_{II})) > - < (x_{III}, f(-x_{III})), (x_{III}, f(-x_{III})) >$$

$$= d((x_I, f(-x_I)), (x_{II}, f(-x_{II})))^2 - d((x_{II}, f(-x_{II})), (x_{III}, f(-x_{III})))^2$$

Hence

$$d((x_I, f(-x_I)), (x_{III}, f(-x_{III})))^2 < d((x_I, f(-x_I)), (x_{II}, f(-x_{II})))^2 + d((x_{II}, f(-x_{II})), (x_{III}, f(-x_{III})))^2$$

Hence adding

$$d((x_I, f(-x_I)), (x_{III}, f(-x_{III})))^2 < d((x_I, f(-x_I)), (x_{II}, f(-x_{II})))^2 + d((x_{II}, f(-x_{II})), (x_{III}, f(-x_{III})))^2$$

$$+2 \cdot d((x_I, f(-x_I)), (x_{II}, f(-x_{II}))) \cdot d((x_{II}, f(-x_{II})), (x_{III}, f(-x_{III})))$$

Hence

$$d((x_I, f(-x_I)), (x_{III}, f(-x_{III})))^2 < (d((x_I, f(-x_I)), (x_{II}, f(-x_{II}))) + d((x_{II}, f(-x_{II})), (x_{III}, f(-x_{III}))))^2$$

Taking square root

$$d((x_I, f(-x_I)), (x_{III}, f(-x_{III}))) < (d((x_I, f(-x_I)), (x_{II}, f(-x_{II}))) + d((x_{II}, f(-x_{II})), (x_{III}, f(-x_{III}))))$$

Hence distance is asymmetric

Note that

$$||g_p(x_I, f(-x_I)) - g_p(x_{II}, f(-x_{II}))||$$

$$= \sqrt{< (x_I, f(-x_I)), (f(-x_I), x_I) > - < (x_{II}, f(-x_{II})), (x_{II}, f(-x_{II})) >}$$

$$= \sqrt{x_I x_I - f(-x_I)f(-x_I) - x_{II}x_+ f(-x_{II})f(-x_{II})}$$

$$\neq$$

$$||g_p(x_{II}, f(-x_{II})) - g_p(x_I, f(-x_I))||$$

$$= \sqrt{< (x_{II}, f(-x_{II})), (x_{II}, f(-x_{II})) > - < (x_I, f(-x_I)), (f(-x_I), x_I) >}$$

$$= \sqrt{x_{II}x_{II} + f(-x_{II})f(-x_{II}) - x_I x_I - f(-x_I)f(-x_I)}$$

Hence the norm not induces a metric (i.e. next property have an exception on Riemann manifolds "Every norm induces a metric"

-) A Riemannian metric is a special case of a metric tensor

Def. Let M be a smooth manifold of dimension n; for instance a surface (in the case $n = 2$) or hypersurface in the Cartesian space $\mathbb{R}^{n+1}$. At each point $p \in M$ there is a vector space T_pM, called the tangent space, consisting of all tangent vectors to the manifold at the point p. A metric tensor at p is a function $g_p(X_p, Y_p)$ which takes as inputs a pair of tangent vectors X_p and Y_p at p, and produces as an output a real number (scalar), so that the following conditions are satisfied:

+) g_p is bilinear. An operation is bilinear if its linear separately in each argument. Thus if U_p, V_p, are two tangent lines at at point p and q of C_1' and C_2' respectively (which could be two open sets), then is bilinear if its linear separately in each argument.

+) g_p is symmetric. A function of two vector arguments is symmetric provided that for all vectors X_p and Y_p,

$$g_p(X_p, Y_p) = g_p(Y_p, X_p)$$

+) g_p is nondegenerate. A bilinear function is non degenerate provided that, for every tangent vector $X_p \neq 0$, the function

$$Y_p \mapsto g_p(X_p, Y_p)$$

obtained by holding X_p constant and allowing Y_p to vary is not identically zero. That is, for every $X_p \neq 0$ there exists a Y_p such that $g_p(X_p, Y_p) \neq 0$

Proof.

Remember that early the exterior product was defined through a system of linear equations that is used to establish the determinant of a matrix nxn where

$$detA = \begin{vmatrix} a & b \\ -e_j & e_i \end{vmatrix} = ae_i + be_j = c \quad ;$$

is the determinant for the matrix of coefficients of the system

$$ax + by = \alpha \quad ;$$

$$-e_j x + e_i y = \beta$$

Now lets take the generalization proved early to establish the value of the entries of the base e_i, e_j and lets take the coefficients of both equations like the independent and a dependent variables of a function $f(x)$; i.e. lets consider the pairs $(x_I, f(x_I))$; $(x_{II}, f(x_{II})) \in T_{P_T}M$ hence with

$$y = m_{||}(x - x_I^*) + f(x_I^*)$$

hence

$$\frac{y}{m_{||}} = (x - x_I^*) + \frac{f(x_I^*)}{m_||} \quad ;$$

$$\frac{y}{m_{||}} - (x - x_I^*) = \frac{f(x_I^*)}{m_||}$$

* the system of linear equations with a, b variables,

*α or β constant terms

$$\frac{f(x_I^*)}{m_{||}} := \alpha \quad (\text{or} \quad \beta)$$

*and χ and γ the unknown coefficients

$$\chi_1 a + \gamma_1 b = \alpha \quad ;$$

$$\chi_2 a + \gamma_2 b = \beta$$

where

$$\gamma = \frac{y_i}{m_{||}};$$

with $y_i = f_(x_i)$

$$\chi = x - x_i^* \quad ;$$

$$\gamma = f_R(\chi)$$

have like matrix of coefficients

$$A = \begin{pmatrix} \chi_1 & f_R(\chi_1) \\ \chi_2 & f_R(\chi_2) \end{pmatrix}$$

and a and b the variables of such system

Now lets clear b from both equations

$$-\frac{\chi_1 a}{f_R(\chi_1)} + \frac{\alpha}{f_R(\chi_1)} = b \quad ;$$

$$-\frac{\chi_2 a}{f_R(\chi_2)} + \frac{\beta}{f_R(\chi_2)} = b \quad ;$$

where

$-\frac{\chi_1}{f_R(\chi_1)} = m_{||1} :=$ slope of line equation 1

$-\frac{\chi_2}{f_R(\chi_2)} = m_{||2} :=$ slope of line equation 2

$\frac{\alpha}{f_R(\chi_1)} := y-$ intercept of equation 1

$\frac{\beta}{f_R(\chi_2)} := y-$intercept of equation 2

$a :=$ independent variable

$b :=$ dependent variable of a

Now lets consider a function g such that

$$g(-\frac{\chi_1 a + \alpha}{f_R(\chi_1)}) = g(b) \quad ;$$

$$g(-\frac{\chi_2 a + \beta}{f_R(\chi_2)}) = g(b) \quad ;$$

With g proportional and additive

$$g(-\frac{\chi_1 a}{f_R(\chi_1)}) + g(\frac{\alpha}{f_R(\chi_1)}) = g(b) \quad ;$$

$$g(-\frac{\chi_2 a}{f_R(\chi_2)}) + g(\frac{\beta}{f_R(\chi_2)}) = g(b) \quad ;$$

If g is linear on a hence

$$a \cdot g(-\frac{\chi_1}{f_R(\chi_1)}) + g(\frac{\alpha}{f_R(\chi_1)}) = g(b) \quad ;$$

$$a \cdot g(-\frac{\chi_2}{f_R(\chi_2)}) + g(\frac{\beta}{f_R(\chi_2)}) = g(b) \quad ;$$

Therefore the system of equations can be classified as bilinear

Now consider a system of n linear equations with n coefficients (i.e. a system of n equations for the n-tuple of entries a_{ij}).

$a_{11}\chi_{11} + a_{12}\chi_{12} + \cdots + a_{1n}\chi_{1n} = \alpha_1$

$a_{21}\chi_{21} + a_{22}\chi_{22} + \cdots + a_{2n}\chi_{2n} = \alpha_2$
$\vdots$
$a_{n1}\chi_{n1} + a_{n2}\chi_{n2} + \cdot + a_{nn}\chi_{nn} = \alpha_n$

where a_{ij} coefficients are known

with determinant

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

Now lets take g_1 and lets apply to the system of equations

$g(a_{11}\chi_{11} + a_{12}\chi_{12} + \cdots + a_{1n}\chi_{1n}) = g(\alpha_1)$

$g(a_{21}\chi_{21} + a_{22}\chi_{22} + \cdots + a_{2n}\chi_{2n}) = g(\alpha_2)$

$\vdots$

$g(a_{n1}\chi_{n1} + a_{n2}\chi_{n2} + \cdot + a_{nn}\chi_{nn}) = g(\alpha_n)$

and consider that g_1 is linear and proportional to χ_{11}, and additive over the other variables

$\chi_{11}g(a_{11}) + g(a_{12}\chi_{12} + \cdots + a_{1n}\chi_{1n}) = g(\alpha_1)$

$\chi_{11}g(a_{21}) + g(a_{22}\chi_{22} + \cdots + a_{2n}\chi_{2n}) = g(\alpha_2)$

$$\vdots$$

$$\chi_{11} g(a_{n1}) + g(a_{n2}\chi_{n2} + \cdots + a_{nn}\chi_{nn}) = g(\alpha_n)$$

Clearing

$$-\chi_{11} g(a_{11}) + g(\alpha_1) = g(a_{12}\chi_{12} + \cdots + a_{1n}\chi_{1n})$$

$$-\chi_{11} g(a_{21}) + g(\alpha_2) = g(a_{22}\chi_{22} + \cdots + a_{2n}\chi_{2n})$$

$$\vdots$$

$$-\chi_{11} g(a_{n1}) + g(\alpha_n) = g(a_{n2}\chi_{n2} + \cdot + a_{nn}\chi_{nn})$$

A set of linear equations is obtained with $g(i1)$ the slope and $g(\alpha_i)$ the $y-$ intercept of a straight line. Establishing the case of a multilinear system.

Note that if the procedure is made extensively, hence the combination of linear equations must be defined to establish a nxn system of equations (n equations of n coefficients).

+) g_p is bilinear

Proof.

With $g_p :=< (\chi_1, f_R(\chi_1)), (\chi_2, f_R(\chi_2)) >= detA$

$$detA = \begin{vmatrix} \chi_1 & f_R(\chi_1) \\ \chi_2 & f_R(\chi_2) \end{vmatrix}$$

with g_1 a function hence applying it to next system of equations

$$g_1(\chi_1 a + f_R(\chi_1) b) = g_1(\alpha_1)$$

$$g_2(\chi_2 a + f_R(\chi_2) b) = g_1(\alpha_2)$$

Hence if g_1 is linear, additive and proportional

$$a g_1(\chi_1) + g_1(f_R(\chi_1) b) = g_1(\alpha_1) \quad ;$$

$$ag_1(\chi_2) + g_1(f_R(\chi_2)b) = g_1(\alpha_2)$$

Clearing

$$-ag_1(\chi_1) + g_1(\alpha_1) = g_1(f_R(\chi_1)b) \quad ;$$

$$-ag_1(\chi_2) + g_1(\alpha_2) = g_1(f_R(\chi_2)b)$$

Where $g_1(\chi_i)$ is the slope and $g_1(\alpha_i)$ is the y - intercept

Hence the system of equations is linear on each equation

Therefore $< (\chi_1, f_R(\chi_1)), (\chi_2, f_R(\chi_2)) >$ is bilinear

+) g_p is symmetric

If

$$abs\left(|det\begin{vmatrix} a & b \\ -e_j & e_i \end{vmatrix}\right) = abs(ae_i + be_j) = abs(-be_j - ae_i) = abs\left(det\begin{vmatrix} -e_j & e_i \\ a & b \end{vmatrix}\right)$$

Hence

$$abs(< (a,b), (-e_j, e_i) >) = abs(< (-e_j, e_i), (a,b) >)$$

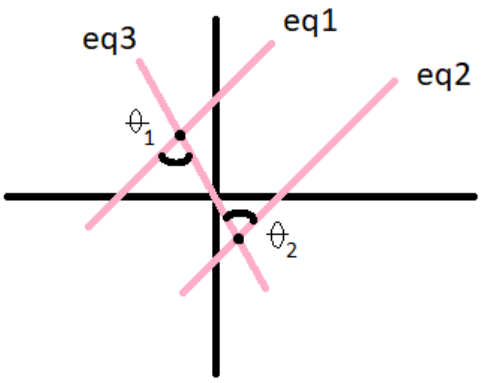
where abs denote absolute value

Hence g_p is symmetric

+) g_p is nondegenerate

Fulfilled like was established early on this text

-) Measure of graph geometry using determinant of linear equation system



Graph of System of equations

Remember dot product definition

$$\vec{A} \cdot \vec{B} = ||\vec{A}|| \cdot ||\vec{B}|| cos\theta$$

Taking early definition about skew symmetric pairing and inner product hence

$$A \cdot R(\theta)B = ||A|| \cdot ||R(\phi)B|| cos\theta$$

with $\phi = 90$

Considering such definition and last graph of system of equations, can be established that two systems of linear equations are defined

$$A_1 = \begin{vmatrix} eq_1 \\ eq_2 \end{vmatrix} \quad \text{and} \quad A_2 = \begin{vmatrix} eq_1 \\ eq_3 \end{vmatrix}$$

with dot product

$$A_{eq1} \cdot A_{eq2} = ||A_{eq1}|| \cdot ||A_{eq2}|| cos\theta_1 \quad ;$$

$$A_{eq1} \cdot A_{eq3} = ||A_{eq1}|| \cdot ||A_{eq3}|| cos\theta_2$$

This means that if θ_i changes (see "graph of system of equations") hence the coordinates of eq_i changes; hence the equation that describes the straight line for the pair $(\chi_i, f_R(\chi_i))$; i.e.

$$eq_i := \chi_i + f_R(\chi_i)b = \alpha_i \quad ;$$

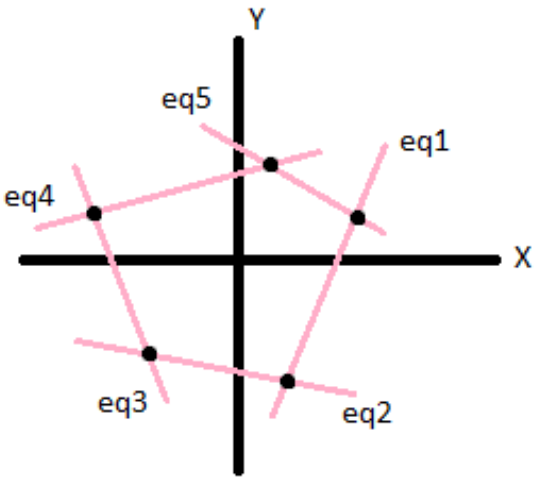
also change and vice versa (i.e. if the coordinates change, hence the angle θ between both lines changes.

Like the value of

$detA = c$

is related with the value of the coordinates of the straight-line equation that defines, and hence is also related with the angle of the dot product that establishes.

Note that with early definitions a graph can be formally established if vertex coordinates are considered.



Graph of C_5

Properties of graphs in Graph Theory will be addressed in a subsequent document to give place to the remainder of the proof about "Hodge Conjecture" in this text.

+) Skew-symmetric pairing is positive-definite

Taking

$$< (\chi_1 1, f_R(\chi_1), (\chi_2, f_R(\chi_2)) >= \chi_1 f_R(\chi_2) - f_R(\chi_1)\chi_2$$

If

1) $\chi_1 f_R(\chi_2) > f_R(\chi_1)\chi_2$

2) $\chi_1 f_R(\chi_2) > 0$ and $f_R(\chi_1)\chi_2 < 0$

3) $\chi_1 f_R(\chi_2) < 0$; $\quad f_R(\chi_1)\chi_2 < 0$ and $\chi_1 f_R(\chi_2) < f_R(\chi_1)\chi_2$

Hence, the skew-symmetric pairing is positive definite.

Hence g_p is Riemannian metric.

Note that like a tangent line can be defined over any open set on the topology of $\mathbb{C}$ hence the equation of such a straight line always is well defined hence the alpha terms (independent terms) for

$$\chi_i a + f_R(\chi_i) b = \alpha_i$$

always are known establishing the y-intercept $\frac{\alpha_i}{f_R(\chi_i)}$; variable with the center of the open set (the circle $C'_{(h_1,h_2)}$).

Taking last definitions about the skew-symmetric pairing and Riemannian metric, hence $(\mathbb{C}, g_p)$ is a Riemannian manifold.

Hence, the symplectic manifold defined early is a Kähler manifold.

Complex viewpoint

Def. A Kähler manifold is a complex manifold X with a Hermitian metric h whose associated 2-form ω is closed. In more detail, h gives a positive definite Hermitian form on the tangent space TX at each point of X, and the 2-form ω is defined by

$$\omega(u,v) = Reh(iu,v) = Imh(u,v)$$

for tangent vectors u and v (where i is the complex number $\sqrt{-1}$). For a Kähler manifold X, the Kähler form ω is a real closed (1,1)-form. A Kähler manifold can also be viewed as a Riemannian manifold, with the Riemannian metric g defined by

$$g(u,v) = Reh(u,v)$$

Equivalently, a Kähler manifold X is a Hermitian manifold of complex dimension n such that for every point p of X, there is a holomorphic coordinate chart around p in which the metric agrees with the standard metric on C^n of order 2 near p. That is, if the chart takes p to 0 in C^n, and the metric is written in these coordinates as

$$h_{ab} = (\frac{\delta}{\delta_{z_a}}, \frac{\delta}{\delta_{z_b}}),$$

then

$$h_{ab} = \delta_{ab} + O(||z||^2)$$

for all $a, b \in \{1, ..., n\}$

Since the 2-form ω is closed, it determines an element in the de Rham cohomology $H^2(X, R)$, known as the Kähler class.

Def. A Hermitian metric on a complex vector bundle E over a smooth manifold M is a smoothly varying positive-definite Hermitian form on each fiber. Such a metric can be viewed as a smooth global section h of the vector bundle $(E \otimes \overline{E})^*$ such that for every point p in M,

$$h_p(\eta, \bar{\zeta}) = \overline{h_p(\zeta, \bar{\eta})}$$

for all

ζ, η in the fiber E_p and

$$h_p(\zeta, \bar{\zeta}) > 0$$

for all nonzero

ζ in E_p.

Def. A complex Hermitian form (also called a symmetric sesquilinear form), is a sesquilinear form

$h : V \times V \to \mathbb{C}$

such that

$h(w, z) = \overline{h(z, w)}$.

Def. A complex skew-Hermitian form (also called an antisymmetric sesquilinear form), is a complex sesquilinear form

$s : V \times V \to \mathbb{C}$

such that

$s(w, z) = -\overline{s(z, w)}$.

Def. (Sesquilinear form)

Over a complex vector space V a map

$$\varphi : V \times V \to \mathbb{C}$$

is sesquilinear if

$$\varphi(x + y, z + w) = \varphi(x, z) + \varphi(x, w) + \varphi(y, z) + \varphi(y, w)$$

$$\varphi(ax, by) = \overline{a} b \, \varphi(x, y)$$

for all $x, y, z, w \in V$ and all $a, b \in \mathbb{C}$. Here, $\overline{a}$ is the complex conjugate of a scalar a.

A complex sesquilinear form can also be viewed as a complex bilinear map

$$\varphi : \overline{V} \times V \to \mathbb{C}$$

where $\overline{V}$ is the complex conjugate vector space to V. By the universal property of tensor products these are in one-to-one correspondence with complex linear maps

$$\overline{V} \otimes V \to \mathbb{C}$$

For a fixed $z \in V$ the map

$$w \mapsto \varphi(z, w)$$

is a linear functional on V (i.e. an element of the dual space V^*). Likewise, the map

$$w \mapsto \varphi(w, z)$$

is a conjugate-linear functional on V.

Given any complex sesquilinear form φ on V we can define a second complex sesquilinear form ψ via the conjugate transpose:

$$\psi(w, z) = \overline{\varphi(z, w)}$$

In general, ψ and φ will be different. If they are the same then φ is said to be Hermitian. If they are negatives of one another, then φ is said to be skew-Hermitian. Every sesquilinear form can be written as a sum of a Hermitian form and a skew-Hermitian form.

Proof.

Remember that the 2-form is

$$detA =< (x_1, f(x_1)), (x_2, f(x_2)) >= (x_1, f(x_1)) \otimes (x_2, f(x_2)) = x_1 f(x_2) x_2 f(x_1)$$

and take into consideration next definition

Def. Let M be a complex manifold. A Hermitian metric is a function

$$h : T_pM \times T_pM \to \mathbb{C}$$

for each point $p \in M$, satisfying:

* **Sesquilinearity**:

with

$x, y, z, w; = x_i \in \mathbb{R}$ with i numbering respectively

Lets take $x_1 + x_2$, $x_3 + x_4 \in \Re$ within $f(x_1, x_2)$, $f(x_3 + x_4) \in \Re$

such that $(x_1 + x_2, f(x_1 + x_2)); (x_3 + x_4, f(x_3 + x_4)) \in T_pM \subseteq \mathbb{C}$

Now lets take the 2- form

**

$$detA =< (x_1 + x_2, f(x_1 + x_2)), (x_3 + x_4, f(x_3 + x_4) >$$

$$= (x_1+x_2, f(x_1+x_2))\otimes(x_3+x_4, f(x_3+x_4)) = (x_1+x_2)f(x_3+x_4)(x_3+x_4)f(x_1+x_2)$$

With $f(x)$ a linear map (additive and homogeneous of degree 1)

$$= (x_1 + x_2)(f(x_3) + f(x_4)) - (x_3 + x_4)(f(x_1) + f(x_2))$$

$$= x_1f(x_3)+x_1f(x_4)+x_2f(x_3)+x_2f(x_4)-x_3f(x_1)-x_3f(x_2)-x_4f(x_1)-x_4f(x_2)$$

$$= x_1f(x_3)-x_3f(x_1)+x_1f(x_4)-x_4f(x_1)+x_2f(x_3)-x_3f(x_2)+x_2f(x_4)-x_4f(x_2)$$

$$= (x_1, f(x_1)) \otimes (x_3, f(x_3)) + (x_1, f(x_1)) \otimes (x_4, f(x_4))$$

$$+(x_2, f(x_2)) \otimes (x_3, f(x_3)) + (x_2, f(x_2)) \otimes (x_4, f(x_4))$$

$$=< x_1, f(x_1)), (x_3, f(x_3)) > + < (x_1, f(x_1)), (x_4, f(x_4)) >$$

$$+ < (x_2, f(x_2)), (x_3, f(x_3)) > + < (x_2, f(x_2)), (x_4, f(x_4)) >$$

$$= detA + detB + detC + detD$$

i.e.

$$\varphi((x+y, f(x+y)), (z+w, f(z+w)) = \varphi(x,z) + \varphi(x,w) + \varphi(y,z) + \varphi(y,w)$$

**

$$detA =< (ax_1, f(ax_1)), (bx_2, f(bx_2)) >= (ax_1, f(ax_1)) \otimes (bx_2, f(bx_2)) = ax_1 f(bx_2) bx_2 f(ax_1)$$

$$= ab(x_1 f(x_2) - x_2 f(x_1)) = ab((x_1, f(x_1) \otimes (x_2, f(x_2)) = ab < (x_1, f(x_1)), (x_2, f(x_2)) >$$

$$= ab \cdot detA$$

i.e.

$$\varphi(ax, by) = ab \cdot \varphi(x, y)$$

with $-a =_A a$

lets define $\bar{a} = -a$

Hence

$$\bar{a}b \cdot \varphi(x,y) < ab \cdot \varphi(x,y) = \varphi(ax,by)$$

Hence

$$\overline{a}b\,\varphi(x,y) =_A \varphi(ax,by)$$

* **Hermitian symmetry**: $h(v,w) = h(\bar{w},\bar{v})$

$$detA = \begin{vmatrix} x_1 & f(x_1) \\ x_2 & f(x_2) \end{vmatrix}$$

$$=< (x_1,f(x_1)),(x_2,f(x_2)) >= (x_1,f(x_1)) \otimes (x_2,f(x_2)) = x_1 f(x_2) x_2 f(x_1)$$

$$detA' = \begin{vmatrix} -x_1 & f(-x_1) \\ -x_2 & f(-x_2) \end{vmatrix}$$

$$=< (-x_1,f(-x_1)),(-x_2,f(-x_2)) >= (-x_1,f(-x_1)) \otimes (-x_2,f(-x_2))$$

$$= -x_1 f(-x_2) - (-1) x_2 f(-x_1)$$

with $f(x)$ pair

$$= x_1 f(x_2) - x_2 f(x_1)$$

Hence $detA = detA'$

Hence ψ is a symmetric sesquilinear form

Hence ψ is Hermitian

If

$$-detA' = -\begin{vmatrix} x_2 & f(x_2) \\ x_1 & f(x_1) \end{vmatrix}$$

$$= - < (x_2, f(x_2)), (x_1, f(x_1)) > \quad = \quad -[(x_2, f(x_2)) \otimes (x_1, f(x_1))]$$

$$= -x_2 f(x_1) + f(x_2) x_1$$

hence ψ is an antisymmetric sesquilinear form

Hence ψ is skew Hermitian

**

with

$$Reh(u, v) = detA = \begin{vmatrix} x_1 & f(x_1) \\ x_2 & f(x_2) \end{vmatrix}$$

$$Imh\overline{(v, u)} = -detA' = -\begin{vmatrix} x_2 & f(x_2) \\ x_1 & f(x_1) \end{vmatrix}$$

hence

$$\omega(u, v) = Reh(u, v) =_A Imh\overline{(v, u)}$$

** $g(u, v) = Reh(u, v)$

Like $g(u, v) :=$ 2-form ω

hence

the Hermitian form is a Riemannian form

** h is a metric

Like g is a metric and

$g = h$

hence

h is a metric

hence also is a bilinear map and a tensor by definition

** holomorphic coordinates

Like the atlas is holomorphic for $\mathbb{C}$

Hence exist an holomorphic coordinate chart around p .

* By early properties the metric is the chart is the same that the standard metric.

Remember that for the pair $(x, f(x))$ when

$\chi_i = x - x^* :=$ la distancia euclidea en 1D

and $\gamma := \frac{f(\chi)}{m_{||}}$

Hence in terms of $m_{||_{z_i}} := \delta_{z_i}$ the slope of the tangent line to the point $(x, f(x))$ and also the derivative on such point the metric is

$$h_{ab} :=< (\chi_{z_a}, \frac{f(\chi_{z_a})}{\delta_{z_a}}), (\chi_{z_b}, \frac{f(\chi_{z_b})}{\delta_{z_b}}) >,$$

in terms of Taylor expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f(n)(a)}{n!}(xa)^n$$

$$h_{ab} :=< (\chi_{z_a}, \frac{\delta_{z_a} + O(f^n(\chi_{z_a}))}{\delta_{z_a}}), (\chi_{z_b}, \frac{\delta_{z_b} + O(f^n(\chi_{z_b}))}{\delta_{z_b}}) >,$$

for all $n \in \{1, ..., N\}$

** With $(x_i, f(x_i))$ on the smooth section

$$\sigma :\in CSB_{(h_1,h_2)} \to E$$

and the metric

$$\sqrt{detA} = \sqrt{< (x_1, f(x_1)), (x_2, f(x_2)) >} = \sqrt{(x_1, f(x_1)) \otimes (x_2, f(x_2))}$$

the domain of h is on

$$B \otimes SB := B \times SB$$

i.e. the fiber

$$F = B \times SB$$

with $(x_1, f(x_1)); (x_2, f(x_2)) \in E = \mathbb{C}$

where and

$$B, SB \subseteq \mathbb{C}$$

i.e. the metric is defined with the vector bundle (E, π)

Hence the metric is Hermitian

Def. A Hermitian manifold is a complex manifold with a Hermitian metric on its holomorphic secant (tangent) bundle. Likewise, an almost Hermitian manifold is an almost complex manifold with a Hermitian metric on its holomorphic secant (tangent) bundle.

On a Hermitian manifold the metric can be written in local holomorphic coordinates (z^α) as

$$h = h_{\alpha\bar{\beta}}\, dz^\alpha \otimes d\bar{z}^\beta$$

where $h_{\alpha\bar{\beta}}$ are the components of a positive-definite Hermitian matrix.

Proof.

Taking $(x_1, f(x_1))$ and $(x_2, f(x_2)) \in CSB_{(h_1,h_2)} \subseteq \mathbb{C}$

where $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are holomorphic coordinates hence

$$h((x_1, f(x_1); (x_2, f(x_2)) = \quad < (x_1, f(x_1)), (x_2, f(x_2)) > \quad = (x_1, f(x_1)) \otimes (x_2, f(x_2))$$

Like $\mathbb{C}$ and almost complex manifold; besides h is hermitian

Hence $\mathbb{C}$ is a Hermitian manifold

Def. The de Rham complex is the cochain complex of differential forms on some smooth manifold M, with the exterior derivative as the differential:

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \to \cdots$$

where $\Omega^0(M)$ is the space of smooth functions on M, $\Omega^1(M)$ is the space of $1-$forms, and so forth. Forms that are the image of other forms under the exterior derivative, plus the constant 0 function in $\Omega^0(M)$, are called exact and forms whose exterior derivative is 0 are called closed (see Closed and exact differential forms); the relationship $d^2 = 0$ then says that exact forms are closed

Def. A chain complex $(A_\bullet, d_\bullet)$ is a sequence of abelian groups or modules $\cdots, A_0, A_1, A_2, \ldots$ connected by homomorphisms (called boundary operators or differentials) $d_n : A_n \to A_{n-1}$, such that the composition of any two consecutive maps is the zero map. Explicitly, the differentials satisfy $d_n \circ d_{n+1} = 0$ for all n, or, concisely, $d^2 = 0$. The complex may be written out as follows:

$$\cdots \xleftarrow{d_0} A_0 \xleftarrow{d_1} A_1 \xleftarrow{d_2} A_2 \xleftarrow{d_3} A_3 \xleftarrow{d_4} A_4 \xleftarrow{d_5} \cdots$$

The cochain complex $(A^\bullet, d^\bullet)$ is the dual notion to a chain complex. It consists of a sequence of abelian groups or modules $\cdots, A^0, A^1, A^2, \ldots$ connected by homomorphisms (co-boundary operators) $d^n : A^n \to A^{n+1} satisfying \mathrm{d}^{n+1} \circ d^n = 0$. The cochain complex may be written out in a similar fashion to the chain complex:

$$\cdots \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} A^3 \xrightarrow{d^3} A^4 \xrightarrow{d^4} \cdots$$

In both cases, the index n is referred to as the degree (or dimension). The difference between chain and cochain complexes is that, in chain complexes, the differentials decrease dimension, whereas in cochain complexes they increase dimension. All the concepts and definitions for chain complexes apply to cochain complexes, except that they will follow this different convention for dimension, and often terms will be given the prefix co-.

Prop. Since the 2-form ω is closed, it determines an element in the De Rham cohomology $H^2(X, R)$, known as the Kähler class.

Def. First Theorem of Isomorphism (Before). Let $f : G \to H$ a group homomorphism. Then exist an isomorphism $\bar{f} : G/(kerf) \to imf$, and then

$$G/(kerf) \cong imf$$

Def. Now. Let $f : G \to H$ a group homomorphism. Then exist $\bar{f} : G/(kerf) \to imf$ bijective, and then

$$G/(kerf) \cong imf$$

Def. Fundamental Theorem of Homomorphism (Before). Given two groups G and H and a group homomorphism $f : G \to H$, let N be a normal subgroup in G and φ the natural surjective homomorphism $\varphi : G \to G/N$ (where G/N is the quotient group of G by N). If N is a subset of $ker(f)$ (where ker represents a kernel) then there exists a unique homomorphism $h : G/N \to H$ such that $f = h \circ \varphi$ where φ is the canonical projection

Def. Fundamental Theorem of Homomorphism on Fiber Bundle (Now). Given two groups G and H and $f : G \to H$, let N be a normal subgroup in G and φ the natural surjective homomorphism $\varphi : G \to G/N$ (where G/N is the dual space of the quotient group of G by N). If $ker(f)$ is a subset of $(G/N)^*$ (where ker represents a kernel and $(G/N)^*$) then there exists a unique $\bar{z} : (G/N)^* \to H$ such that $f = \bar{z} \circ \varphi$ where φ is the canonical projection

Def. Normal group. A subgroup N of a group G its called normal subgroup of G if lateral classes by left and right defined by any $g \in G$ agree; i.e. $gN = Ng$; $\forall g \in G$.

Def. Inner Automorphism. If G is a group and g is an element of G (alternatively, if G is a ring, and g is a unit) then the function

$\varphi_g : G \to G$

$\varphi_g(x) := g^{-1}xg$

is called (right) conjugation by g . For all $x_1, x_2 \in G$,
$\varphi_g(x_1x_2) = g^{-1}x_1x_2g = g^{-1}x_1(gg^{-1})x_2g = (g^{-1}x_1g)(g^{-1}x_2g) = \varphi_g(x_1)\varphi_g(x_2)$,

Def. Conjugation. On a group, the relation can be defined as

$\forall a, b \in G : a \sim b \iff b = g^{-1} \circ a \circ g$

for some $g \in G$

such that a is related with b if a and b are conjugated. The relation defined is an equivalence relation.

Then, the conjugated elements of an element "a" form a class, called class of conjugation of "a"

$$[a] = \{x^{-1}ax : x \in G\}$$

The conjugation by a element in particular of the group is an isomorphism of group.

Def. Group Action. A group action $(G, *)$ over the set X is an application $\phi : G \times X \to X$ that fulfills next two conditions:

1. $\forall x \in X : \phi(e, x) = x$, where e is the neutral element of the group.

2. $\forall x \in X : g, h \in G, \phi(g * h, x) = \phi(g, \phi(h, x))$

In such case this means that G acts over X, and that X is a $G-$set

Def. Quotient Group. Let N be a normal subgroup of a group G. Define the set G/N to be the set of all left cosets of N in G. That is, $G/N = \{aN : a \in G\}$.

Def. Chain Homomorphism Lets consider the structures A, B, C, D and its homomorphisms

$$f : A \to B; \quad g : B \to C; \quad h : C \to D$$

Then the "chain homomorphism" is the composition

$$h \circ g \circ f : A \to D$$

where such composition is an homomorphism because the composition of homomorphism is an homomorphism

Def. Iterated Composition of Operators. Let T be an operator; then the iterated composition is

$$T^n = \underbrace{T \circ T \circ ... \circ T}_{n}$$

Def. Operator. Let V and W two spaces. An operator is an application (function)

$$T : V \to W$$

that assigns to each element of $v \in V$ an element $T(v) \in W$.

Def. Linear Operator. When $V = W$ is a vectorial space over a field K and T fulfills

$$T(av + bw) = aT(v) + bT(w) \quad ; \forall v, w \in V, a, b \in K$$

then T is linear operator.

Def. Derivative Operator. Formally

$$D : F \to F; \quad D(f)(x) = f'(x)$$

where F is the space of differentiable functions

* Linearity

$$D(af + bg) = aD(f) + bd(g)$$

with a, b are constants

* Iteration

$$D^n(f) = f^{(n)}(x)$$

where n establish the n-esim derivative

Def. Homology Group. Next condition defines a chain

$$d^i \circ d^{i+1} = 0$$

This guarantee that *****

$$H^i = \frac{ker(d^i)}{im(d^{n+1})}$$

Def. Cohomology Group. Next condition defines a cochain

$$d^{i+1} \circ d^i = 0; \quad \forall i$$

This guarantee that $Im(d^{i-1}) \subset Ker(d^i)$

i.e.

$$H^i = \frac{ker(d^i)}{im(d^{i-1})}$$

is a cohomology group

Proof.

Let take next definition

$$\psi(x, f(x)) := g \circ Prod(x, f(x))$$

and lets consider Prod function

$$Prod(x, f(x)) : \mathbb{C} \to \Re$$

Like $\mathbb{C}$ is a vector space hence is an abelian group with addition i.e. $G =: (\mathbb{C}, +)$

Now lets define the normal subgroup N of $G := (\mathbb{C}, +)$ and the definition of lateral class

Def. Let G a group and N a subgroup of G. For an element of $g \in G$

Left lateral class : $gN = \{gn : n \in N\}$

Right lateral class: $Ng = \{ng : n \in N\}$

Proof.

Now lets define the lateral class of G

Like G is a group under addition i.e.

$$gN := g + n$$

with $(x_g, f(x_g)) \in G$ and $(x_n, f(x_n)) \in N$ hence

$$(x_g, f(x_g)) + (x_n, f(x_n)) := [(x_g, f(x_g)), (x_n, f(x_n))] := L_{[(x_g, f(x_g)), (x_n, f(x_n))]}$$

establish the left lateral class

while

$$Ng := n + g$$

$$(x_n, f(x_n)) + (x_g, f(x_g)) := [(x_n, f(x_n)), (x_g, f(x_g))] := L_{[(x_n, f(x_n)), (x_g, f(x_g))]}$$

establish the right lateral class

With

$$L_{[(x_g,f(x_g)),(x_n,f(x_n))]} := L_l$$

and

$$L_{[(x_n,f(x_n)),(x_g,f(x_g))]} := L_r$$

Now remember that the sense of each line is defined like

$$\mathfrak{G}(x(t), y(t)) = \frac{\mathfrak{h}(x(t), y(t)) - \mathfrak{h}_{min}}{\mathfrak{h}_{max} - \mathfrak{h}_{min}} \cdot f_v$$

with $\mathfrak{h}(x(t), y(t))$ the value of the random variable and

$$\theta = \frac{G_i - G_j}{G_{max} - G_{min}} \cdot 360$$

For $(x_g, f(x_g))$

$$G(x_g(t), y_g(t)) = \frac{\mathfrak{h}(x_g(t), y_g(t)) - \mathfrak{h}_{min}}{\mathfrak{h}_{max} - \mathfrak{h}_{min}} \cdot f_v$$

and $(x_n, f(x_n))$

$$G(x_h(t), y_h(t)) = \frac{\mathfrak{h}(x_n(t), y_n(t)) - \mathfrak{h}_{min}}{\mathfrak{h}_{max} - \mathfrak{h}_{min}} \cdot f_v$$

Now lets consider

$$\theta_{gn} = \frac{G_{(x_g,f(x_g))} - G_{(x_n,f(x_n))}}{G_{max} - G_{min}} \cdot 360$$

and

$$\theta_{ng} = \frac{G_{(x_n,f(x_n))} - G_{(x_g,f(x_g))}}{G_{max} - G_{min}} \cdot 360$$

where

G_{max} := max value of $G(x(t), y(t))$ with $(x(t), y(t)) \in G$

G_{min} := min value of $G(x(t), y(t))$ with $(x(t), y(t)) \in G$

Now lets define the set

$$L_r = \{[(x_n, f(x_n)), (x_g, f(x_g))] \in N \times G \quad | \quad (x_h, f(x_h)) \in N \subseteq \mathbb{C} \text{ AND}$$

$$(x_g, f(x_g)) \in G \subseteq \mathbb{C}; \text{ with } m = \frac{f(x_g) - f(x_h)}{x_g - x_h}$$

$$\text{with } \theta_{ng} := \text{sense} \quad ;$$

$$\text{and } d[(x_n, f(x_n)), (x_g, f(x_g))] = d_{ng}\}$$

and

$$L_l = \{[(x_g, f(x_g)), (x_n, f(x_n))] \in G \times N \quad | \quad (x_n, f(x_n)) \in N \subseteq \mathbb{C} \text{ AND } (x_g, f(x_g)) \in G \subseteq \mathbb{C};$$

$$\text{with } m = \frac{f(x_n) - f(x_g)}{x_n - x_g} \quad ;$$

$$\text{with } \theta_{gn} := \text{sense} \quad ;$$

$$\text{and } d[(x_g, f(x_g)), (x_n, f(x_n))] = d_{gn}\}$$

Hence

$$L_r \neq L_l$$

i.e.

$$gN \neq Ng$$

while taking

$$[(x_g(t), y_g(t)), (x_n(t), y_n(t))] := (x_g(t), y_g(t)) \in G \text{ AND } (x_n(t), y_n(t)) \in N$$

$$[(x_n(t), y_n(t), (x_g(t), y_g(t)] := (x_n(t), y_n(t)) \in N \text{ AND } (x_g(t), y_g(t)) \in G$$

Now lets establish (like in ref[2, 3, 4, 5]) next definition about equality

(Hyp.) Prop.

$X \times Y =_{AND} Y \times X$ [*i.e.* $=_{AND}$:= $\equiv$ or logically equivalent]

Like

p AND q is logically equivalent to q AND p

i.e.

p AND $q \equiv q$ AND p

hence

$x \in X$ AND $y \in Y$ is logically equivalent to $y \in Y$ AND $x \in X$

i.e.

$x \in X$ AND $y \in Y \quad \equiv \quad y \in Y$ AND $x \in X$

i.e.

$(x, y) \in X \times Y \quad \equiv \quad (y, x) \in Y \times X$

with

$=_{AND}$:= $\equiv$

Like early proposition is true hence

$L_l =_{AND} L_r$

hence

$gN =_{AND} Ng$

Hence

N is "**AND** normal" in G

Remember that $Prod(x, f(x))$ is a homomorphism

And lets define the quotient group of G by N

Like the quotient group is defined like $G/N = \{aN : a \in G\}$

Hence with addition

$aN := (x_a, f(x_a)) + [(x_n, f(x_n))] = [(x_a, f(x_a)), (x_n, f(x_n))] = L$

The product of two lateral classes

$$L \cdot L = |L| \cdot |L| cos(\theta)$$

i.e.

$$G/N := L_{G/N} = \{[(x_a, f(x_a)), (x_n, f(x_n))] \in G \times N \quad |$$

$$(x_a, f(x_a)) \in G \subseteq \Re^2 \text{ AND } (x_n, f(x_n)) \in N \subseteq \mathbb{C};$$

$$\text{with } m = \frac{f(x_a) - f(x_n)}{x_a - x_n} \quad ;$$

$$\text{with } \theta_{an} := \text{sense} \quad ;$$

$$\text{and } d[(x_a, f(x_a)), (x_n, f(x_n))] = d_{an}\}$$

Now consider the dual space of G/N

$$(G/N)^* = \{q(t) \in Q(t) \quad | \quad d[(x_a, f(x_a)), (x_n, f(x_n))] = d_{an} = q(t) \text{ and}$$

$$R(\theta)\vec{v} = q(t)R(\theta')\vec{v} \quad ;$$

$$\text{fulfills for } (x_a, f(x_a)) \text{ and } (x_n, f(x_n)) \quad \}$$

Now remember that **PROD** function is an homomorphism (ref[2,3,4]) and lets prove that is surjective

Proof.

$$\forall x \cdot f(x) \in Im(\mathbf{PROD}); \quad \exists (x, f(x)) \in \mathcal{C} : \mathbf{PROD}(x, f(x)) = x \cdot f(x)$$

Hence **PROD** is a surjective homomorphism

*

Now lets establish next property about homomorphism and derivative

First lets take the function z such that dz is its derivative hence

$$dz(a) = lim_{h \to 0} \frac{z(a+h) - z(a)}{h}$$

$$= \frac{z(a+h)}{h} + (-\frac{z(a)}{h})$$

Early was established that

$$= \frac{z(a+\delta) - z(a-\delta)}{(a+\delta) - (a-\delta)}$$

Now considers that $(G/N)^* = Q(t) \subseteq \Re$ is a group under product

Taking $a = q(t) = x \cdot f(x)$ and substituting on last relation about T

$$dz(q(t)) = \frac{z(q(t)+\delta) - z(q(t)-\delta)}{(q(t)+\delta) - (q(t)-\delta)}$$

$$dz(x \cdot f(x)) = \frac{z(x \cdot f(x)+\delta) - z(x \cdot f(x)-\delta)}{(x \cdot f(x)+\delta) - (x \cdot f(x)-\delta)}$$

$$dz(x \cdot f(x)) = \frac{z(x \cdot f(x)+\delta)}{(x \cdot f(x)+\delta) - (x \cdot f(x)-\delta)} + (-\frac{z(x \cdot f(x)-\delta)}{(x \cdot f(x)+\delta) - (x \cdot f(x)-\delta)})$$

with

$$h = (x \cdot f(x)+\delta) - (x \cdot f(x)-\delta)$$

Hence

$$dz(x \cdot f(x)) = \frac{z(x \cdot f(x)+\delta) - z(x \cdot f(x)-\delta)}{h})$$

$$dz(Prod(x, f(x))) = \frac{z(Prod(x, f(x))+\delta) - z(Prod(x, f(x))-\delta)}{h})$$

$$dz(Prod(x, f(x))) = \frac{z(Prod(x, f(x))+\delta) - z(Prod(x, f(x))-\delta)}{h})$$

Hence with $\psi(x, f(x)) := z \circ Prod(x, f(x))$

$$d\psi(x, f(x)) = \frac{z(Prod(x, f(x))+\delta) - z(Prod(x, f(x))-\delta)}{h})$$

Now lets obtain the kernel of the differential operator "**d**"

Lets take

$$dz : (G/N)^* \to Im(dz)$$

with

$$\mathbf{Prod}(x, f(x)) \in (G/N)^*$$

and

$$dz \circ \mathbf{Prod}(x, f(x)) \in Im(dz)$$

and

$$d : V_d \to dz \in W_{dz}$$

with

$$z \circ (Prod(x, f(x))) \in V_{z \circ Prod(x, f(x))}$$

and

$$d(z \circ Prod(x, f(x)) \in W_{d(z \circ Prod(x, f(x)))}$$

Now lets consider kernel definition

Def.

$$f : G \to H$$

$$kerf = \{g \in G : f(g) = e_H\}$$

Proof.

Hence

$$ker \quad d = \{z \in V_{z \circ Prod(x,f(x))} : dz = e_{W_{dz}} \quad \text{and} \quad Prod(x, f(x)) \in (G/N)^*\}$$

with $e_{W_{dz}}$ multiplicative identity

Lets consider

$$(G/N)^*_{ker\ d} \subseteq (G/N)^*$$

such that

$$(gn)_{ker\ d} \in (G/N)_{ker\ d} \quad ;$$

where

$$(G/N)_{ker\ d} := L_{ker\ d} = \{[(x_a, f(x_a)), (x'_n, f(x'_n))] \in G \times N_{ker\ d} \quad |$$

$$(x_a, f(x_a)) \in G \subseteq \Re^2 \text{ AND } (x'_n, f(x'_n)) \in N_{ker\ d} \subseteq \mathbb{C};$$

$$\text{with } m = \frac{f(x_a) - f(x'_n)}{x_a - x'_n} \quad ;$$

$$\text{with } \theta_{an'} := \text{sense} \quad ;$$

$$\text{and } d[(x_a, f(x_a)), (x'_n, f(x'_n))] = d_{an'}\}$$

with

$$g \in G$$

and

$$n' \in N_{ker\ d}$$

and consider its dual space such that

$$q(t)_{ker\ d} \in (G/N)^*_{ker\ d}$$

and

$$\psi : G \times X \to X$$

*

$\Psi(x_e, x) = g \circ Prod(x_e, x) = g(x_e \circ x)$

$\forall x \in X$

where x_e is the neutral element of $\Re$

and

*

with $x_g, x_h \in G$

$$\Psi(x_g \circ x_h, x) = g(Prod((x_g \circ x_h) \circ x)$$

$$= g(x_g \circ x_h \circ x) = g(x_g, Prod(x_h, x))$$

$$= \Psi(x_g, Prod(x_h, x))$$

if g = dz = e

$$\Psi(x_g \circ x_h, x) = e(Prod((x_g \circ x_h) \circ x)$$

$$= e(x_g \circ x_h \circ x) = e(x_g \circ Prod(x_h, x))$$

$$= e(x_g, Prod(x_h, x)) = e(x_g, e \circ Prod(x_h, x))$$

$$= \Psi(x_g, \Psi(x_h, x))$$

hence $G := \Re$ acts over $\Re$; i.e. is a group action

with $g(t) \in (G/N)^*$

$$dz(q(t))_{ker\ d} = e_{W_{dz}}$$

Now lets take

$$g \in (G/N)^*_{ker\ d}$$

and lets define $\bar{z}$

$$\bar{z}_g : (G/N)^*_{ker\ d} \to (G/N)^*_{ker\ d}$$

$$\bar{z}_g(q(t)_{ker\ d}) := g^{-1} q(t)_{ker\ d} g$$

a function over $(G/N)^*_{ker\ d}$

such that for all

$$q(t)_{1_{ker\ d}}, \quad q(t)_{2_{ker\ d}} \in (G/N)^*_{ker\ d}$$

$$\bar{z}_g(q(t)_{1_{ker\ d}} q(t)_{2_{ker\ d}}) = g^{-1} q(t)_{1_{ker\ d}} q(t)_{2_{ker\ d}} g$$

$$= g^{-1} q(t)_{1_{ker\ d}} (g g^{-1}) q(t)_{2_{ker\ d}} g$$

$$= (g^{-1}q(t)_{1_{ker\ d}}g)(g^{-1})q(t)_{2_{ker\ d}}g)$$

$$= \bar{z}_g(q(t)_{1_{ker\ d}})\bar{z}_g(q(t)_{2_{ker\ d}})$$

such that $\bar{z}$ is an Inner Automorphism

Like $(G/N)^*_{ker\ d}$ is a group under product hence $\bar{z}_g(q(t)_{ker\ d}) \in (G/N)^*_{ker\ d}$

Hence

$$q(t)_{ker\ d} \sim \bar{z}_g(q(t)_{ker\ d})$$

With equivalence class

$$[q(t)_{ker\ d}] = \{g^{-1}q(t)_{ker\ d}g : g \in (G/N)^*_{ker\ d}\}$$

Hence

$$q(t)_{ker\ d} \sim \bar{z}_g(q(t)_{ker\ d})$$

is an equivalence relation

Like $\bar{z}$ is a function hence if is biyective and considering that

$$(G/N)^*_{ker\ d} \subseteq (G/N)^* \subseteq Q(t) \subseteq \Re$$

have an isomorphism if

* $\Re$ is considering a vector space

* $\Re$ is field and only is isomorphic to itself

* $\Re$ is an additive group $(\Re, +)$

hence if $(G/N)^*_{ker\ d}$ is considered one of the early conditions exist an isomorphism

$$\bar{z} : (G/N)^*_{ker\ d} \to Im\bar{z}$$

i.e.

$$(G/N)^*_{ker\ d} \cong Im\bar{z}$$

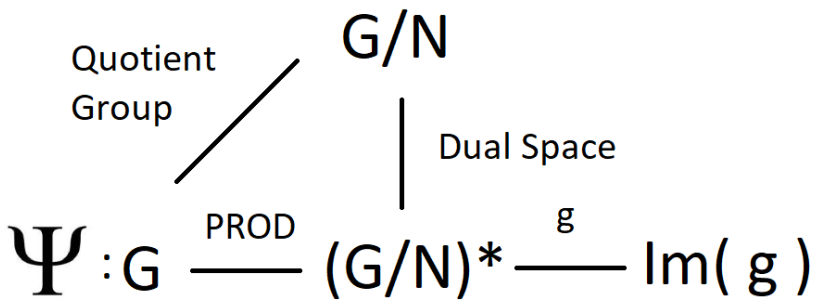
Hence the "First Theorem of Isomorphism" is fulfilled

Like $\bar{z}$ is unique by kernel condition $N_{ker\ d} \subseteq N$

Hence the "Fundamental Theorem of Homomorphism on Fiber Bundle" is fulfilled

*

Lets take next diagram



First Theorem of Homomorphism for Fiber Bundles

Where

$G := \mathbb{C}$

$N := N \subseteq \mathbb{C}$

Im g := subset of $\Re$

With $\Psi(x, f(x)) := z \circ Prod(x, f(x))$

Lets consider now next code in Python where

$f(x_g) :=$ is a function

$x_g :=$ independent variable

the product $PROD(x_g, f(x_g)) = x_g \cdot f(x_g)$ is calculated

and the condition

$d = \sqrt{(x_g - x_n)^2 + (f(x_g) - f(x_n))^2}$

is established for $d = x_g \cdot f(x_g)$

to find the point $(x_n, f(x_n))$ evaluating $f(x_n)$ for different points of x_n in steps of $\delta_2 = 0.01$ for 20 point of x_g separated by $\delta_1 = 0.1$

Question on Copilot Microsoft to obtain the code in Python

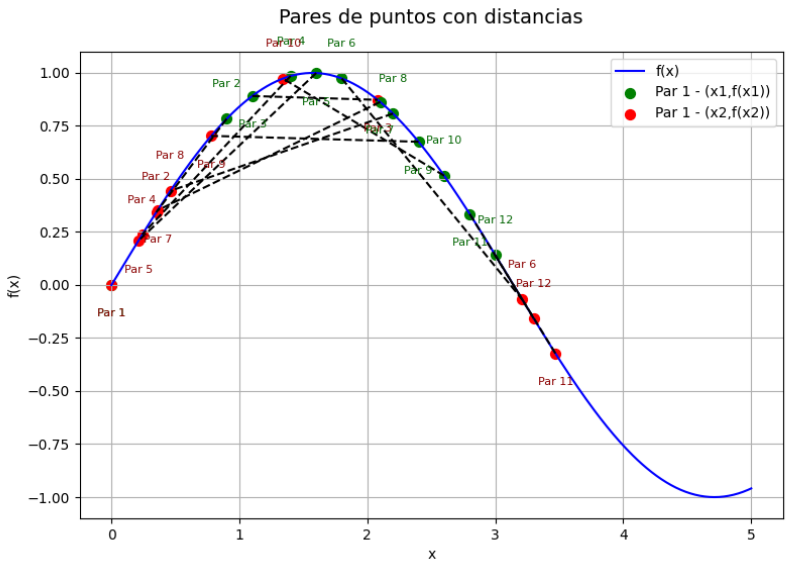
$-o-$

```
import numpy as np
import math
import matplotlib.pyplot as plt

# ------------------------------
# 1) Define la funci n f(x)
# ------------------------------
def f(x):
    return np.sin(x)   # <-- c mbiala por la funci n que quieras

# ------------------------------
# 2) Par metros
# ------------------------------
a, b = 0, 5        # intervalo para x1
delta1 = 0.1       # paso entre puntos iniciales
delta2 = 0.01      # paso de b squeda para x2
tol = 1e-3         # tolerancia para igualdad de distancias

# ------------------------------
# 3) C lculos
# ------------------------------
```

```
x1_values = np.arange(a, b, delta1)
pares = []

for x1 in x1_values:
    y1 = f(x1)
    target_d = x1 * y1   # distancia objetivo

    for x2 in np.arange(a, b, delta2):
        y2 = f(x2)
        d = math.sqrt((x1 - x2)**2 + (y1 - y2)**2)
        if abs(d - target_d) < tol:
            pares.append(((x1, y1), (x2, y2)))
            print(f"x1={x1:.2f}, y1={y1:.2f} | target d={target_d:.3f} | x2={x2:
            break  # detener b squeda al encontrar el primer x2 v lido

# ------------------------------
# 4) Visualizaci n
# ------------------------------
xs = np.linspace(a, b, 400)
ys = f(xs)

plt.figure(figsize=(9,6))
plt.plot(xs, ys, label="f(x)", color="blue")

# Etiquetar cada par alternando arriba/abajo y con colores distintos
for idx, ((x1,y1),(x2,y2)) in enumerate(pares, start=1):
    etiqueta = f"Par {idx}"
    plt.plot([x1,x2],[y1,y2],"k--") # l nea negra discontinua entre los puntos

   # Primer punto en verde
    plt.scatter(x1, y1, color="green", s=50, label=f"{etiqueta} - (x1,f(x1))" if
    plt.text(x1, y1+0.15 if idx%2==0 else y1-0.15, etiqueta, fontsize=8, color="

    # Segundo punto en rojo
    plt.scatter(x2, y2, color="red", s=50, label=f"{etiqueta} - (x2,f(x2))" if id
    plt.text(x2, y2+0.15 if idx%2==0 else y2-0.15, etiqueta, fontsize=8, color="


plt.xlabel("x")
plt.ylabel("f(x)")
plt.title("Pares de puntos con distancias", pad=20, fontsize=14, loc="center")
plt.legend()
plt.grid(True)
plt.show()
```

$-o-$

This program shows next graph



Graph of $f(x) = sin(x)$ for $(x_g, f(x_g))$ "green point" and $(x_n, f(x_n))$ "red point"

and next table of results

Table 1. Pairs $(x_n, f(x_n))$ found with distance $d_{gn} = x_g \cdot f(x_g) = \sqrt{(x_g - x_n)^2 + (f(x_g) - f(x_n))^2}$

x1=0.00, y1=0.00	target d=0.000	x2=0.00, y2=0.00
x1=0.90, y1=0.78	target d=0.705	x2=0.35, y2=0.34
x1=1.10, y1=0.89	target d=0.980	x2=2.08, y2=0.87
x1=1.40, y1=0.99	target d=1.380	x2=0.24, y2=0.24
x1=1.60, y1=1.00	target d=1.599	x2=0.21, y2=0.21
x1=1.80, y1=0.97	target d=1.753	x2=3.21, y2=−0.07
x1=2.10, y1=0.86	target d=1.813	x2=0.36, y2=0.35
x1=2.20, y1=0.81	target d=1.779	x2=0.46, y2=0.44
x1=2.40, y1=0.68	target d=1.621	x2=0.78, y2=0.70
x1=2.60, y1=0.52	target d=1.340	x2=1.34, y2=0.97
x1=2.80, y1=0.33	target d=0.938	x2=3.47, y2=−0.32
x1=3.00, y1=0.14	target d=0.423	x2=3.30, y2=−0.16

With

$$d : V_z \to W_{dz}$$

where

$z \in V_z$

and

$dz \in W_{dz}$

Like $f(x)$ is the n-derivative of Z hence

$f(x) = d^n z$

On $q(t) = x_g \cdot f(x_g) \in (G/N)^* \subseteq Q(t)$ where $(x_g, f(x_g)) \in G := \mathbb{C}$ the derivative is

$$d^n z \circ PROD(x_g, f(x_g)) = d^n z(x_g \cdot f(x_g)) = d^n z(x_g \cdot f(x_g)) = d^n z \circ PROD(x_g, f(x_g))$$

i.e. with $d^n z(x_g \cdot f(x_g)) = m_{\|}$ hence

$$d^n z(x_g \cdot f(x_g)) = \frac{d^n z(x_g \cdot f(x_g) + \delta) - d^n z(x_g \cdot f(x_g) - \delta)}{(x_g \cdot f(x_g) + \delta) - (x_g \cdot f(x_g) - \delta)}$$

establishing the interval $[x_g f(x_g) - \delta, x_g f(x_g) + \delta]$ hence the integration could be defined with the derivative equals to $m_{\|}$ like

$$\int_{x_g \cdot f(x_g) - \delta}^{x_g \cdot f(x_g) + \delta} dz^n(x) dx = d^{n-1} z(x)$$

where $d^n z(x) = f(x)$ besides to be an integration of $f(x)$

* Definition of Interval of Integration

Lets consider first

Def. Let X be a set and Σ a $\sigma-$algebra over X, defining subsets of X that are "measurable". A set function μ from Σ to the extended real number line, that is, the real number line together with new (so-called infinite) values $+\infty$ and $-\infty$, respectively greater and lower than all other (so-called finite) elements, is called a measure if the following conditions hold:

$\mu(\varnothing) = 0$

Non-negativity: For all $E \in \Sigma, \quad \mu(E) \geq 0$

Countable additivity (or σ-additivity): For all countable collections

$\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ,

$\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$

If at least one set E has finite measure, then the requirement $\mu(\varnothing) = 0$ is met automatically due to countable additivity:

$\mu(E) = \mu(E \cup \varnothing) = \mu(E) + \mu(\varnothing)$, and therefore

$\mu(\varnothing) = 0$

Note that any sum involving $+\infty$ will equal $+\infty$, that is, $a + \infty = +\infty$ for all a in the extended reals.

If the condition of non-negativity is dropped, and $\mu(E)$ only ever equals one of $+\infty$, $-\infty$, i.e. no two distinct sets have measures $+\infty$, $-\infty$, respectively, then μ is called a signed measure.

The pair (X, Σ) is called a measurable space, and the members of Σ are called measurable sets.

A triple (X, Σ, μ) is called a measure space. A probability measure is a measure with total measure one–that is, $\mu(X) = 1$. A probability space is a measure space with a probability measure.

Proof.

Lets consider first the definition of σ-algebra

Def. Let X be some set, and let $P(X)$ represent its power set, the set of all subsets of X. Then a subset $\Sigma \subseteq P(X)$ is called a σ- **algebra** if it satisfies the following three properties:

1. X is in Σ

2. Σ is closed under complementation: If some set A is in Σ, then so is its complement, $X \setminus A$

3. Σ is closed under countable unions: If $A_1, A_2, A_3, ...$ are in Σ, the so is $A = A_1 \cup A_2 \cup A_3 \cup ...$

Elements of $\sigma-$ algebra are called measurable sets.

A function between two measurable spaces is called a measurable function if the preimage of every measurable set is measurable.

With

$$I = [x - \delta, x + \delta] = \{x \in \Re | x - \delta \leq x \leq x + \delta\} \quad ;$$

$$|I| = (x + \delta) - (x - \delta) = 2\delta$$

Hence

$$|B| = |I_1|$$

taking this definitions hence

$$P(I) = \{E_i | E_i \subseteq [x - \delta, x + \delta]\}$$

with cardinality

$$|P([x - \delta, x + \delta))| = 2^c$$

Now lets consider

$$\Sigma \subseteq P([x - \delta, x + \delta])$$

and next conditions

1.- $[x - \delta, x + \delta] \in \Sigma$

2.- Σ is closed under complementation; If $E_i \in \Sigma \rightarrow [x-\delta, x+\delta]/E_i \in \Sigma$

3.- Σ is closed under countable unions; If $E_1, E_2, E_3, ... \in \Sigma \rightarrow E = E_1 \cup E_2 \cup E_3 \cup ...$

Hence Σ is a σ-algebra

Now lets consider

$$\mu(E_i) = f(E_i)$$

with next properties

1) $\mu(\phi) = 0$

2) $\forall E_i \in \Sigma; \mu(E_i) \geq 0$

3) For all countable collections $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint sets in Σ

$$\mu(\bigcup_{k=1}^{\infty} E_k) = \Sigma_{k=1}^{\infty} \mu(E_k)$$

Hence

$([x-\delta, x+\delta], \Sigma)$ is called a measurable space and the members of Σ are called measurable sets

$([x-\delta, x+\delta]), \Sigma, \mu)$ is called a measured space

Now consider the property 3 (sigma closed under countable unions) and "**Choice Axiom** ") to establish next set

Now lets take the Choice Axiom

Axiom. For any set X of nonempty sets, there exists a choice function f that is defined on X and maps each set of X to an element of that set

$$\forall X : \emptyset \notin X \rightarrow \exists (f : X \rightarrow \bigcup X) : \forall A \in X, f(A) \in A$$

With

$$X = E_{i\,i=1}^{\infty} \quad ;$$

$$A = E_i$$

and

$$f := f_{Choice}$$

hence

$$\forall \{E_i\}_{i=1}^{\infty} : \emptyset \notin \{E_i\}_{i=1}^{\infty} \rightarrow \exists (f_{Choice} : \{E_i\}_{i=1}^{\infty} \rightarrow \bigcup_{i=1}^{\infty} E_i) : \forall E_i \in \{E_i\}_{i=1}^{\infty}, f_{Choice}(E_i) \in E_i$$

by the

Def. Intermediate Value Theorem. If f is continuous on [a,b] then f([a,b]) is an interval

hence

with $f_{Choice}(E_i) = Im(f_{Choice}(E_i))$

i.e.

$$f(I) = \{f(x) | x \in I\}$$

Hence by " **Choice Axiom** " with

$$E = [x - \delta, x + \delta] = \bigcup_{i=1}^{\infty} E_i$$

where

$$E_i = [x - \delta_i, x + \delta_i]$$

and

$$\delta_i = \delta_0(\#steep(i))$$

such that $E := E_0 = [x - \delta, x + \delta]$ and $\delta = \delta_0$ and the number of steep is the number iteration that is defined to establish an interval

Now consider next two properties

1.- ***Choice Axiom - Intersection**

If f_{Choice} is applied over an element $x \in [x - \delta, x + \delta] = E_0 = \bigcup_{i=1}^{\infty}\{E_i\}_{i=1}^{\infty}$ with $E_i = [x - \delta_i, x + \delta_i]$

Hence

$$x \in \bigcap_{i=1}^{\infty} E_i$$

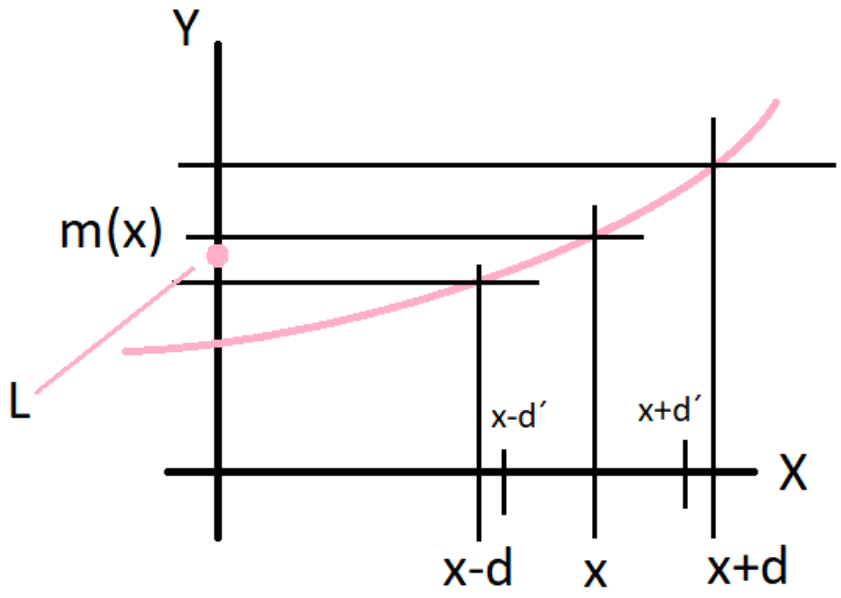
Hence next postulate is true

$$\forall\{E_i\}_{i=1}^{\infty} : \phi \in \{E_i\}_{i=1}^{\infty} \rightarrow \exists(f : \{E_i\}_{i=1}^{\infty} \rightarrow \bigcap_{i=1}^{\infty}\{E_i\}_{i=1}^{\infty} : \forall E_i \in \{E_i\}_{i=1}^{\infty}; f(E_i) \in E_i$$

Hence

Choice Axiom is valid for Intersection over any interval

2.- * $\exists \quad \delta_i = \delta_0(\#steep(i))$



$x - d : x - \delta, x - d'$ another interval like δ_i and δ_{i+1} respectively

Taking $\epsilon - \delta$ limit definition

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in \Re)(0 < |x - p| < \delta) \rightarrow |f(x) - L| < \epsilon$$

hence with

$$f(x) := \mu(x \in E) \quad ;$$

$$lim_{x+\delta \rightarrow x} \quad \mu(x \in [x - \delta, x + \delta]) = L_\mu \quad ;$$

$$\delta_i = \delta_0(\#steep(i))$$

Hence

$$(\forall \epsilon > 0)(\exists d_i > 0) \text{ s.t. } (\forall i \in I)(\forall x \in \{E_i\}_{i=1}^{\infty})(0 < |x - p| < \delta_i = \delta_0(\#steep(i))) \rightarrow |\mu(x) - L_\mu| < \epsilon$$

******

** Computer code on Python about "Choice Axiom" for Intersection

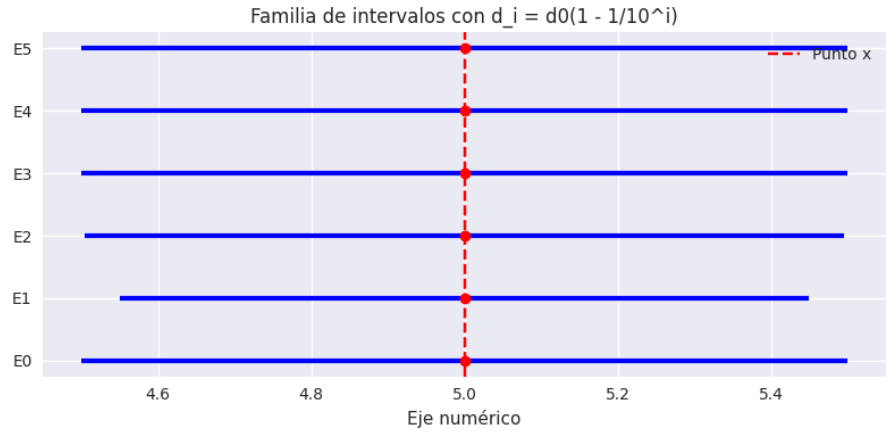
```
# Generando intervalos con d_i = d_0(1 - 1/10^i), seleccionando punto x,
# verificando pertenencia y graficando resultados

import matplotlib.pyplot as plt
import numpy as np

# Par metros
x = 5.0      # Punto central
d0 = 0.5     # Valor que define el intervalo m s grande (E0)
n = 5        # N mero de intervalos

# Generar intervalos
intervalos = []
d_values = []

# Intervalo m s grande E0
intervalos.append((x - d0, x + d0))
d_values.append(d0)

# Intervalos Ei m s peque os
for i in range(1, n + 1):
    d_i = d0 * (1 - 1 / (10 ** i))   # F rmula modificada
    d_values.append(d_i)
    intervalo = (x - d_i, x + d_i)
    intervalos.append(intervalo)

# Seleccionar el punto x (centro de los intervalos)
x_seleccionado = x

# Verificar si todos los intervalos contienen el punto
todos_contienen = all(inf <= x_seleccionado <= sup for (inf, sup) in intervalos)

# Resultados en consola
print(f"Se seleccion  el punto x = {x_seleccionado}")
print(f"Intervalo m s grande E0: (x-d0, x+d0) = ({x - d0:.6f}, {x + d0:.6f})")
print(f"Se consideraron {len(intervalos)} intervalos en total.")
for i, inter in enumerate(intervalos):
    print(f"  E{i}: ({inter[0]:.6f}, {inter[1]:.6f}) con d = {d_values[i]:.6f}")
if todos_contienen:
```

```
    print("Todos los intervalos contienen el punto x.")
else:
    print("Alg n intervalo no contiene el punto x.")

# Gr fica: Intervalos en la l nea num rica
plt.style.use('seaborn-v0_8')
fig, ax = plt.subplots(figsize=(8, 4))
for idx, (inf, sup) in enumerate(intervalos):
    ax.hlines(y=idx + 1, xmin=inf, xmax=sup, color='blue', linewidth=3)
    ax.plot(x_seleccionado, idx + 1, 'ro')  # punto x en rojo

ax.axvline(x=x_seleccionado, color='red', linestyle='--', label='Punto x')
ax.set_yticks(range(1, len(intervalos) + 1))
ax.set_yticklabels([f"E{i}" for i in range(len(intervalos))])
ax.set_xlabel("Eje num rico")
ax.set_title("Familia de intervalos con d_i = d0(1 - 1/10^i)")
ax.legend()
plt.tight_layout()
plt.show()
```

Results

```
Se seleccion  el punto x = 5.0
Intervalo m s grande E0: (x-d0, x+d0) = (4.500000, 5.500000)
Se consideraron 6 intervalos en total.
  E0: (4.500000, 5.500000) con d = 0.500000
  E1: (4.550000, 5.450000) con d = 0.450000
  E2: (4.505000, 5.495000) con d = 0.495000
  E3: (4.500500, 5.499500) con d = 0.499500
  E4: (4.500050, 5.499950) con d = 0.499950
  E5: (4.500005, 5.499995) con d = 0.499995
Todos los intervalos contienen el punto x.


[Execution Finished]
```

Code in Python on Microsoft Copilot Bing
Compiled code on matplotlib online



Now lets consider the "Choice Axiom" and next relation between derivative and integration of a function

$\Psi(x, f(x)) := g \circ Prod(x, f(x)) = g(x \cdot f(x))$

**

$$f(x) \xrightarrow{\text{Integral}} g(x)$$

Lets take the "Choice Axiom" for Intersection and establish that

$$x \in \bigcap_{i=1}^{\infty} \{E_i\}_{i=1}^{\infty} \quad \text{such that} \quad E_0 \supseteq E_1 \supseteq E_2 \supseteq E_3 \supseteq ...$$

where

$$E_0 = [x - \delta \quad , \quad x + \delta]$$

Now consider the points $(x-\delta, f(x-\delta))$ and $(x+\delta, f(x+\delta))$ and the joining line

$$P_1(x) = f(x-\delta) + \frac{f(x+\delta) - f(x-\delta)}{(x+\delta) - (x-\delta)}(x - (x-\delta))$$

$$P_1(x) = f(x-\delta) + \frac{f(x+\delta) - f(x-\delta)}{2\delta}(\delta)$$

$$P_1(x) = f(x-\delta) + \frac{f(x+\delta) - f(x-\delta)}{2}$$

Following trapezoidal rule hence

$$\int_{x-\delta}^{x+\delta} P_1(x)dx = ((x+\delta) - (x-\delta))[f(x-\delta) + \frac{f(x+\delta) - f(x-\delta)}{2}]$$

$$= 2\delta \cdot [f(x-\delta) + \frac{f(x+\delta) - f(x-\delta)}{2}]$$

Besides like

$$\int_{x-\delta}^{x+\delta} f(x)dx = g(x)$$

Hence

$\exists \quad g(x)$ the primitive of $f(x)$

**

$$f(x) \xleftarrow{\text{Derivative}} g(x)$$

By Choice Axion on Intersection

$$x \in \bigcap_{i=1}^{\infty} \{E_i\}_{i=0}^{\infty}$$

with

$$E_0 = [x-\delta, x+\delta]$$

Like

$$P_1(x) = L_{secant}[(x-\delta, g(x-\delta)), (x+\delta, g(x+\delta))]$$

Hence $\exists \quad m_{\|}$ and tangent to the point $(x, g(x))$

Hence $\exists\, dg(x) = f(x)$ the derivative of $g(x)$

**

Relation between $(\mathbb{C}, +)$ groups with derivative and integration

Lets consider the diagram of the function $\Psi(x, f(x))$ by pairs

$$(x_g, f(x_g)) \in G \to [(x_g, f(x_g)), (x_n, f(x_n))] \in G/N$$

$$[(x_g, f(x_g)), (x_n, f(x_n))] \in G/N \to q(t) \in (G/N)^*$$

$$(x, f(x)) \in G \to (x \cdot f(x)) \in (G/N)^*$$

$$(x \cdot f(x)) \in (G/N)^* \to g(x \cdot f(x)) \in Im(g)$$

with

$$f(x) \Longleftrightarrow g(x)$$

related by integral to the left (f to g) and derivative to the right (g to f)

By "Choice Axiom on Intersection" for Integration

$$d_{\to} := \begin{cases} x \in \bigcap_{i=1}^{\infty} \{E_i\}_{i=0}^{\infty}; \\ E_0 \supseteq E_1 \supseteq E_2 \supseteq ...; \\ E_0 = [x-\delta, x+\delta]; \\ \delta_i = \delta_0(\#steep(i)); \\ P_1(x) = f(x-\delta) + \frac{f(x+\delta)-f(x-\delta)}{2}; \\ \int_{x-\delta}^{x+\delta} P_1(x)dx = 2\delta \cdot [f(x-\delta) + \frac{f(x+\delta)-f(x-\delta)}{2}] \end{cases}$$

While for derivative

$$d_{\leftarrow} := \begin{cases} x \in \bigcap_{i=1}^{\infty} \{E_i\}_{i=0}^{\infty} & ; \\ E_0 \supseteq E_1 \supseteq E_2 \supseteq ... & ; \\ E_0 = [x-\delta, x+\delta] & ; \\ \delta_i = \delta_0(\#steep(i)) & ; \\ P_1(x) = g(x-\delta) + \frac{g(x+\delta)-g(x-\delta)}{2} & ; \\ P_1(x) = L_{Secant}[(x-\delta, g(x-\delta), (x+\delta, g(x+\delta)] & ; \\ \text{with } m_{\parallel} \text{ then } L_{tan} := y = (x-x_g) \cdot m_{||} + g(x_g) & ; \\ L_{tan} \text{ tangent to the point } (x_g, g(x_g)) \end{cases}$$

Like $m_{\parallel}$ of L_{tan} is a derivative hence $d(x_g \wedge f(x_g))$ the derivative defined early is the exterior derivative of such pair.

Between any pair of groups $(\mathbb{C}, +)$

Note that

$$(x \cdot f(x), g(x \cdot f(x))) \in (G/N)^* \times Im(g(x \cdot f(x))) = G'$$

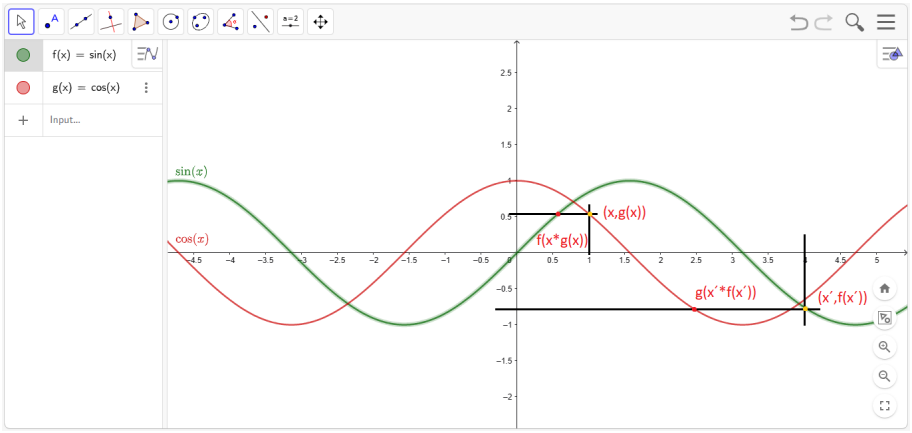
where G' is the set of higher or lower grade that must be defined to establish the chain of complex like is defined on diagram about Ψ function

Note also that the connection between both diagrams; that define the integration fo f to g and the derivation of g to f is

$$(x, f(x)) \in G \xrightarrow{d_{\rightarrow}} (x \cdot f(x), g'(x \cdot f(x)) \in G'$$

$$(x, f(x)) \in G \xleftarrow{d_{\leftarrow}} (x \cdot f(x), g'(x \cdot f(x)) \in G'$$

Example



Note that

$d_n : A_n ß A_{n1}$, such that the composition of any two consecutive maps is the zero map if

- The derivative of a constant function is zero

** Next are shown the code in python for the function $f_{Choice} := d_{\rightarrow}$

$$d_{\rightarrow} : (\{E_i\}_{i=0}^{\infty}; +) \mapsto (Im(\int_{x-\delta}^{x+\delta} P_1(x)dx); \cdot)$$

Code in Python for the function $f_{Choice} := d_{\rightarrow}$ on Copilot Microsoft Compiled Code in matplotlib

–0–

```
import numpy as np
import matplotlib.pyplot as plt

# Definir la funci n f(x)
def f(x):
    return x**2   # puedes cambiarla por np.sin(x), np.exp(x), etc.

# Par metros
```

```
x = 2.0   # punto central
d0 = 1.0  # intervalo m s grande (E0)
n = 5     # n mero de intervalos Ei

# Intervalo principal E0
E0 = (x - d0, x + d0)

# Familia de intervalos Ei con la nueva f rmula
d_values = [d0]
intervalos = []
for i in range(1, n+1):
    d_i = d0 * (1 - 1 / (10 ** i))   # F rmula modificada
    d_values.append(d_i)
    intervalos.append((x - d_i, x + d_i))

# El intervalo m s grande
max_d = d0

# Valores de la funci n en los extremos
f_left = f(x - max_d)
f_right = f(x + max_d)

# Definir polinomio P1(x) como recta entre (x-d0, f(x-d0)) y (x+d0, f(x+d0))
X_poly = np.linspace(x - max_d, x + max_d, 200)
# pendiente de la recta
m = (f_right - f_left) / (2*max_d)
# ecuaci n de la recta
Y_poly = f_left + m * (X_poly - (x - max_d))

# Integral aproximada con P1(x) (trapecio)
Integral_P1 = (f_left + f_right) / 2 * (2*max_d)

print(f"Punto x = {x}")
print(f"Intervalo m s grande E0: d0 = {d0}")
print(f"Valor f(x-d0) = {f_left}, f(x+d0) = {f_right}")
print(f"Integral aproximada con P1(x): {Integral_P1}")
print("Familia de intervalos:")
print(f"  E0: ({E0[0]:.6f}, {E0[1]:.6f}) con d0 = {d0:.6f}")
for i, inter in enumerate(intervalos, start=1):
    print(f"  E{i}: ({inter[0]:.6f}, {inter[1]:.6f}) con d = {d_values[i]:.6f}")

# -----------------------
# Gr fica
# -----------------------
plt.style.use('seaborn-v0_8')
fig, ax = plt.subplots(figsize=(9, 6))
```

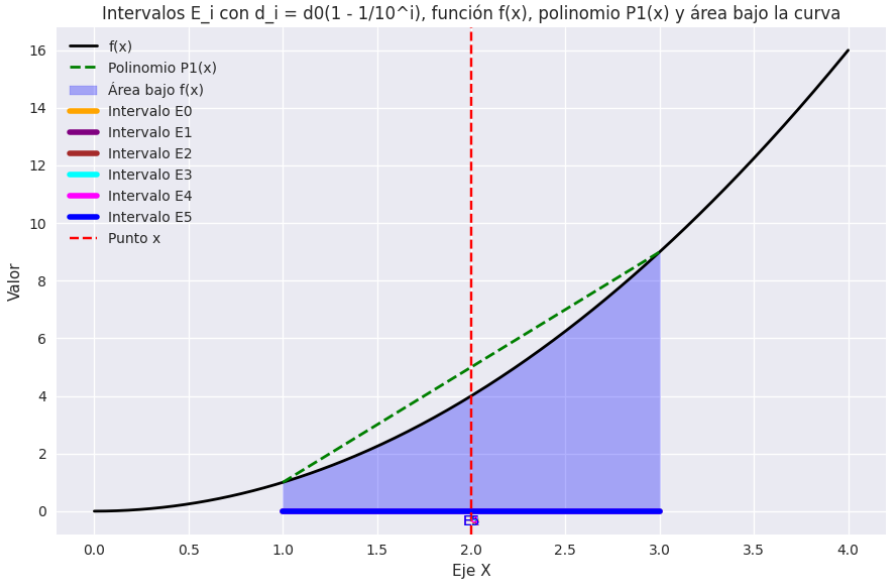
```
# Graficar la funci n f(x)
X_func = np.linspace(x - max_d - 1, x + max_d + 1, 400)
Y_func = f(X_func)
ax.plot(X_func, Y_func, 'k-', linewidth=2, label='f(x)')

# Graficar el polinomio P1(x) como recta
ax.plot(X_poly, Y_poly, 'g--', linewidth=2, label='Polinomio P1(x)')

#  rea  bajo la curva f(x) en el intervalo m s grande
X_area = np.linspace(x - max_d, x + max_d, 400)
Y_area = f(X_area)
ax.fill_between(X_area, Y_area, color='blue', alpha=0.3, label=' rea  bajo f(x)'

# Dibujar intervalos sobre el eje X con colores distintos
colors = ['orange', 'purple', 'brown', 'cyan', 'magenta', 'blue']
for idx, (inf, sup) in enumerate([E0] + intervalos):
    ax.plot([inf, sup], [0, 0], color=colors[idx % len(colors)], linewidth=4,
            label=f"Intervalo E{idx}")
    ax.text((inf+sup)/2, -0.5, f"E{idx}", ha='center', color=colors[idx % len(co

# L nea vertical en x
ax.axvline(x=x, color='red', linestyle='--', label='Punto x')

# Etiquetas
ax.set_xlabel("Eje X")
ax.set_ylabel("Valor")
ax.set_title("Intervalos E_i con d_i = d0(1 - 1/10^i), funci n f(x), polinomio
ax.legend(loc='upper left')
plt.tight_layout()
plt.show()
```

–o–

Results

```
Punto x = 2.0
Intervalo m s grande E0: d0 = 1.0
Valor f(x-d0) = 1.0, f(x+d0) = 9.0
Integral aproximada con P1(x): 10.0
Familia de intervalos:
  E0: (1.000000, 3.000000) con d0 = 1.000000
```

```
E1: (1.100000, 2.900000) con d = 0.900000
E2: (1.010000, 2.990000) con d = 0.990000
E3: (1.001000, 2.999000) con d = 0.999000
E4: (1.000100, 2.999900) con d = 0.999900
E5: (1.000010, 2.999990) con d = 0.999990

[Execution Finished]
```



Pseudocode (in spanish) for f_{Choice} function

Pseudocode to define function $f_{Choice} := d_{\rightarrow}$ on copilot Microsoft Bing

$-o-$

INICIO

```
Definir funci n f(x):
    f(x) = x^2   # puede cambiarse por otra funci n

Definir par metros:
    x = 2.0          # punto central
    d0 = 1.0         # intervalo m s grande (E0)
```

```
    n = 5            # n mero de intervalos Ei

Definir intervalo principal:
    E0 = (x - d0, x + d0)

Generar familia de intervalos Ei:
    Crear lista d_values = [d0]
    Crear lista intervalos = []
    PARA i desde 1 hasta n:
        d_i = d0 * (1 - 1 / (10^i))   # f rmula modificada
        Agregar d_i a d_values
        Definir Ei = (x - d_i, x + d_i)
        Agregar Ei a intervalos
    FIN PARA

Calcular extremos del intervalo m s grande:
    f_left = f(x - d0)
    f_right = f(x + d0)

Definir polinomio lineal P1(x):
    Crear vector X_poly entre (x - d0) y (x + d0)
    Calcular pendiente m = (f_right - f_left) / (2*d0)
    Calcular Y_poly = f_left + m * (X_poly - (x - d0))

Calcular integral aproximada con P1(x) (regla del trapecio):
    Integral_P1 = (f_left + f_right) / 2 * (2*d0)

Imprimir resultados:
    - Punto x
    - Intervalo m s grande E0
    - Valores f(x-d0), f(x+d0)
    - Integral aproximada con P1(x)
    - Lista de intervalos E0, E1, ..., En con sus valores de d

Graficar:
    - Funci n f(x) en un rango amplio
    - Polinomio P1(x) como recta discontinua
    -  rea  bajo f(x) en el intervalo E0
    - Intervalos E0 y Ei sobre el eje X con colores distintos
    - L nea vertical en x
    - Etiquetas y leyenda

FIN
```

$-o-$

Pseudocode (in english) for the function f_{Choice}

Pseudocode to define the function $f_{Choice} := d_{\rightarrow}$ on Microsoft Copilot Bing

$-o-$

START

```
Define function f(x):
    f(x) = x^2   # can be changed to np.sin(x), np.exp(x), etc.

Define parameters:
    x = 2.0          # central point
    d0 = 1.0         # largest interval (E0)
    n = 5            # number of additional intervals Ei

Define main interval:
    E0 = (x - d0, x + d0)

Generate family of intervals Ei:
    Create list d_values = [d0]
    Create list intervals = []
    FOR i from 1 to n:
        d_i = d0 * (1 - 1 / (10^i))   # modified formula
        Add d_i to d_values
        Define Ei = (x - d_i, x + d_i)
        Add Ei to intervals
    END FOR

Compute function values at extremes of the largest interval:
    f_left = f(x - d0)
    f_right = f(x + d0)

Define linear polynomial P1(x):
    Create vector X_poly between (x - d0) and (x + d0)
    Compute slope m = (f_right - f_left) / (2*d0)
    Compute Y_poly = f_left + m * (X_poly - (x - d0))

Compute approximate integral with P1(x) (trapezoid rule):
```

```
    Integral_P1 = (f_left + f_right) / 2 * (2*d0)

Print results:
    - Point x
    - Largest interval E0
    - Values f(x-d0), f(x+d0)
    - Approximate integral with P1(x)
    - List of intervals E0, E1, ..., En with their d values

Plot:
    - Function f(x) over a wide range
    - Polynomial P1(x) as dashed line
    - Area under f(x) in interval E0
    - Intervals E0 and Ei on the X-axis with different colors
    - Vertical line at x
    - Labels and legend

END
```

$-o-$

** Next are shown the code in Python for the function $f_{Cochoice} := d_{\leftarrow}$

$$d_{\leftarrow} : (\{E_i\}_{i=1}^{\infty}; +) \mapsto (Im\big(L_{tan(x,f(x))}\big); \cdot)$$

Code in Python for the function $f_{Cochoice} := d_{\leftarrow}$ Copilot Microsoft Bing

Compiled Code on matplotlib

$-0-$

```
import numpy as np
import matplotlib.pyplot as plt

# 1. Definir la funci n f(x)
def f(x):
    return np.sin(x)   # puedes cambiarla si quieres
```

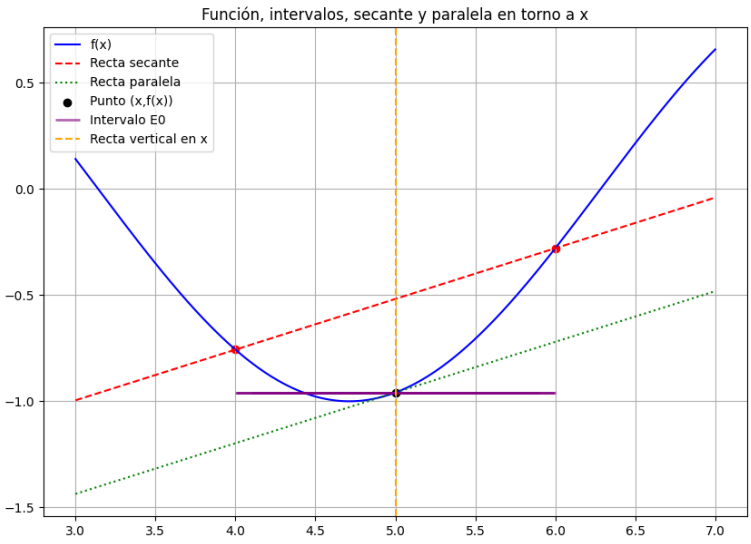
```
# 2. Par metros iniciales
x = 5.0   # punto central
d0 = 1.0  # intervalo m s grande (E0)
n = 3     # n mero de intervalos Ei

# 3. Construcci n de intervalos
intervalos = [(x - d0, x + d0)]  # E0
for i in range(1, n + 1):
    d_i = d0 * (1 - 1 / (10 ** i))  # f rmula modificada
    intervalos.append((x - d_i, x + d_i))

# 4. Verificaci n: x est en todos los intervalos?
todos_contienen = all(inf <= x <= sup for (inf, sup) in intervalos)
print(" x pertenece a todos los intervalos?", todos_contienen)

# 5. Polinomio P1(x) definido por el intervalo m s grande
P1_x = f(x - d0) + (f(x + d0) - f(x - d0)) / 2
print("P1(x) =", P1_x)

# 6. Recta secante entre (x-d0, f(x-d0)) y (x+d0, f(x+d0))
m_secante = (f(x + d0) - f(x - d0)) / (2 * d0)
b_secante = f(x - d0) - m_secante * (x - d0)

# 7. Distancias geom tricas
d1 = np.sqrt((2 * d0) ** 2 + (f(x + d0) - f(x - d0)) ** 2)
d2 = np.sqrt((d0) ** 2 + (f(x) - f(x - d0)) ** 2)
d3 = np.sqrt((d0) ** 2 + (f(x + d0) - f(x)) ** 2)

# 8.  ngulos
theta = np.arccos((2 * d0) / d1)
phi = np.arccos(d0 / d2)
omega = np.arccos(d0 / d3)

# Definiciones adicionales
d1d2 = theta - phi
eta = np.deg2rad(180 - 90 - np.rad2deg(omega))
d1d3 = np.deg2rad(180 - 90 - np.rad2deg(eta) - np.rad2deg(theta))

# 9. Pendiente paralela seg n f rmula
m_paralell = ((f(x) + d3 * np.cos(d1d3) * np.sin(theta)) - (f(x) - d2 * np.cos(d
              ((x + d3 * np.cos(d1d3) * np.cos(theta)) - (x - d2 * np.cos(d1d2) *

# 10. Gr fica
X_vals = np.linspace(x - 2 * d0, x + 2 * d0, 200)
Y_secante = m_secante * X_vals + b_secante
Y_paralell = m_paralell * (X_vals - x) + f(x)
```

```
plt.figure(figsize=(10, 7))

# Funci n original
X_plot = np.linspace(x - 2 * d0, x + 2 * d0, 400)
plt.plot(X_plot, f(X_plot), label="f(x)", color="blue")

# Secante y paralela
plt.plot(X_vals, Y_secante, '--', label="Recta secante", color="red")
plt.plot(X_vals, Y_paralell, ':', label="Recta paralela", color="green")

# Puntos extremos y central
plt.scatter([x - d0, x + d0], [f(x - d0), f(x + d0)], color="red")
plt.scatter([x], [f(x)], color="black", label="Punto (x,f(x))")

# Dibujar intervalos como segmentos horizontales
for idx, (inf, sup) in enumerate(intervalos):
    plt.hlines(y=f(x), xmin=inf, xmax=sup, color="purple", linewidth=2, alpha=0.
               label="Intervalo E{}".format(idx) if idx == 0 else None)

# Recta vertical que pasa por (x,f(x))
plt.axvline(x=x, color="orange", linestyle="--", label="Recta vertical en x")

plt.title("Funci n, intervalos, secante y paralela en torno a x")
plt.legend()
plt.grid(True)
plt.show()

print("Pendiente secante =", m_secante)
print("Pendiente paralela =", m_paralell)
```

–0–

Results

```
 x  pertenece a todos los intervalos? True
P1(x) = -0.518108996753427
Pendiente secante = 0.23869349855450117
Pendiente paralela = 0.23869349855450084


[Execution Finished]
```



Pseudoconde (in spanish) for the function $f_{Cochoice} := d_{\leftarrow}$

–0–

```
FUNCI N f_Cochoice(f, x, d0, n):

    # Paso 1: Definir intervalo principal E0
    E0 = (x - d0, x + d0)

    # Paso 2: Construir familia de intervalos Ei con f rmula modificada
    Crear lista intervalos = [E0]
    PARA i DESDE 1 HASTA n:
        d_i = d0 * (1 - 1 / (10^i))
        Ei = (x - d_i, x + d_i)
        AGREGAR Ei A intervalos
    FIN PARA

    # Paso 3: Verificar si x pertenece a todos los intervalos
    todos_contienen = VERIFICAR si x      cada intervalo

    # Paso 4: Calcular valor medio del polinomio P1(x)
```

```
    P1_x = (f(x - d0) + f(x + d0)) / 2

    # Paso 5: Calcular recta secante entre extremos del intervalo E0
    m_secante = (f(x + d0) - f(x - d0)) / (2 * d0)
    b_secante = f(x - d0) - m_secante * (x - d0)

    # Paso 6: Calcular distancias geom tricas
    d1 = distancia entre extremos de E0
    d2 = distancia entre (x, f(x)) y (x - d0, f(x - d0))
    d3 = distancia entre (x, f(x)) y (x + d0, f(x + d0))

    # Paso 7: Calcular  ngulos  entre vectores
    theta =  ngulo  entre extremos de E0
    phi =  ngulo  entre x y extremo izquierdo
    omega =  ngulo  entre x y extremo derecho

    # Paso 8: Calcular diferencias angulares
    d1d2 = theta - phi
    eta = 180  - 90  - omega
    d1d3 = 180  - 90  - eta - theta

    # Paso 9: Calcular pendiente de recta paralela
    m_paralell = f rmula basada en d2, d3, theta, d1d2, d1d3

    # Paso 10: Graficar resultados
    - Graficar f(x) en rango amplio
    - Graficar recta secante
    - Graficar recta paralela
    - Dibujar intervalos Ei como segmentos horizontales
    - Marcar punto (x, f(x))
    - Dibujar l nea vertical en x

    # Paso 11: Imprimir resultados
    -  x  pertenece a todos los intervalos?
    - Valor de P1(x)
    - Pendiente secante
    - Pendiente paralela

FIN FUNCI N
```

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Pseudocode (in English) for the function $f_{Cochoice} := d_{\leftarrow}$ on Copilot Microsoft Bing

```
FUNCTION f_Cochoice(f, x, d0, n):

    # Step 1: Define main interval E0
    E0 = (x - d0, x + d0)

    # Step 2: Build family of intervals Ei using formula
    intervals = [E0]
    FOR i FROM 1 TO n:
        d_i = d0 * (1 - 1 / (10^i))
        Ei = (x - d_i, x + d_i)
        ADD Ei TO intervals
    END FOR

    # Step 3: Check if x belongs to all intervals
    belongs = CHECK if x is inside every interval

    # Step 4: Compute polynomial P1(x) midpoint value
    P1_x = f(x - d0) + (f(x + d0) - f(x - d0)) / 2

    # Step 5: Compute secant line between (x-d0, f(x-d0)) and (x+d0, f(x+d0))
    slope_secant = (f(x + d0) - f(x - d0)) / (2 * d0)
    intercept_secant = f(x - d0) - slope_secant * (x - d0)

    # Step 6: Compute geometric distances
    d1 = distance between (x-d0, f(x-d0)) and (x+d0, f(x+d0))
    d2 = distance between (x, f(x)) and (x-d0, f(x-d0))
    d3 = distance between (x, f(x)) and (x+d0, f(x+d0))

    # Step 7: Compute angles
    theta = angle at secant endpoints
    phi   = angle at left side
    omega = angle at right side

    # Step 8: Additional angle definitions
    d1d2 = theta - phi
    eta  = 180   - 90   - omega
    d1d3 = 180   - 90   - eta - theta

    # Step 9: Compute slope of parallel line
    slope_parallel = formula using d2, d3, theta, d1d2, d1d3

    # Step 10: Plot results
        - Plot f(x) over a wide range
```

```
        - Plot secant line
        - Plot parallel line
        - Draw intervals Ei as horizontal segments
        - Mark point (x, f(x))
        - Draw vertical line at x

    # Step 11: Print results
        - Does x belong to all intervals?
        - Value of P1(x)
        - Secant slope
        - Parallel slope
```

END FUNCTION

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Now lets define the kernel of the function $f_{Choice} := d_{\rightarrow}$

Code in Python for to define the kernel of the function $f_{Choice} := d_{\rightarrow}$

Compiled code to define the kernel of the function $f_{Choice} = d_{\rightarrow}$ in matplotlib

–0–

```
import numpy as np
import matplotlib.pyplot as plt

# Definir la funci n
def f(x):
    return x**2   # puedes cambiarla por np.sin(x), np.exp(x), etc.

# Regla del trapecio
def trapezoidal_integral(f, a, b, n=1000):
    x = np.linspace(a, b, n+1)
    y = f(x)
    h = (b - a) / n
    return (h/2) * (y[0] + 2*np.sum(y[1:-1]) + y[-1])

# Buscar delta con b squeda binaria
def find_delta(f, x, target_area=1, tol=1e-6):
    delta_min, delta_max = 0.0, 10.0  # rango inicial
```

```
    while (delta_max - delta_min) > tol:
        delta_mid = (delta_min + delta_max) / 2
        area = trapezoidal_integral(f, x - delta_mid, x + delta_mid)
        if area < target_area:
            delta_min = delta_mid
        else:
            delta_max = delta_mid
    delta = (delta_min + delta_max) / 2
    return delta, (x - delta, x + delta)

# Verificar que x est  en todos los intervalos
def check_intervals(x, d0, n=5):
    intervals = []
    # Intervalo E0
    intervals.append((x - d0, x + d0))
    # Intervalos Ei con la nueva f rmula
    for i in range(1, n+1):
        d_i = d0 * (1 - 1/(10**i))
        intervals.append((x - d_i, x + d_i))
    # Verificaci n
    belongs = all(interval[0] <= x <= interval[1] for interval in intervals)
    return intervals, belongs

# Ejemplo con f(x)=x^2 y x=2
x0 = 2.0
delta, interval = find_delta(f, x0, target_area=1)

# d0 es el intervalo m s grande
d0 = delta
n = 5  # n mero de intervalos Ei

intervals, belongs = check_intervals(x0, d0, n)

print(f"d0 (intervalo m s grande): {d0:.6f}")
print(f"Intervalo E0: (x-d0, x+d0) = ({x0-d0:.6f}, {x0+d0:.6f})")
print("Intervalos Ei (m s peque os que E0):")
for i, inter in enumerate(intervals[1:], start=1):
    d_i = d0 * (1 - 1/(10**i))
    print(f"  E{i}: (x-d_i, x+d_i) = ({x0-d_i:.6f}, {x0+d_i:.6f}) con d_i={d_i:.
print(f" x  pertenece a todos los intervalos?: {belongs}")

# Graficar la funci n y el  rea
X = np.linspace(x0-2, x0+2, 400)
Y = f(X)

plt.plot(X, Y, label="f(x)")
```

```
plt.axvline(x0, color="red", linestyle="--", label="x0")
plt.fill_between(X, Y, where=(X>=interval[0]) & (X<=interval[1]), alpha=0.3, col
for i, inter in enumerate(intervals[1:], start=1):
    plt.axvline(inter[0], color="blue", linestyle=":", alpha=0.5)
    plt.axvline(inter[1], color="blue", linestyle=":", alpha=0.5)
plt.title(" rea  bajo la curva = 1 con E_i     E0")
plt.legend()
plt.show()
```

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Results

```
d0 (intervalo m s grande): 0.124838
Intervalo E0: (x-d0, x+d0) = (1.875162, 2.124838)
Intervalos Ei (m s peque os que E0):
  E1: (x-d_i, x+d_i) = (1.887646, 2.112354) con d_i=0.112354
  E2: (x-d_i, x+d_i) = (1.876410, 2.123590) con d_i=0.123590
  E3: (x-d_i, x+d_i) = (1.875287, 2.124713) con d_i=0.124713
  E4: (x-d_i, x+d_i) = (1.875174, 2.124826) con d_i=0.124826
  E5: (x-d_i, x+d_i) = (1.875163, 2.124837) con d_i=0.124837
 x  pertenece a todos los intervalos?: True


[Execution Finished]
```

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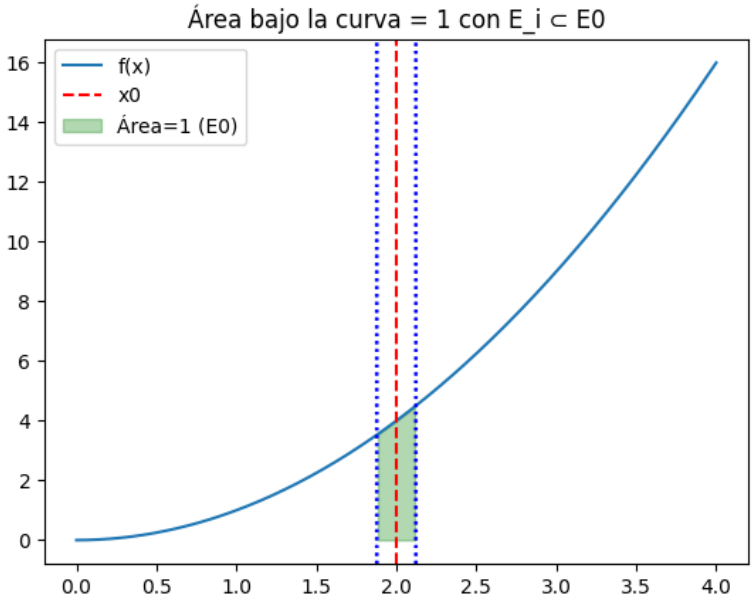

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Pseudocode (in Spanish) to define the kernel of the function $f_{Choice} := d_{\rightarrow}$

```
FUNCIN kernel_f_Choice(f, x0, area_objetivo, n):

    # Paso 1: Encontrar delta (d0) tal que el  rea  bajo f(x) sea igual a area_o
    LLAMAR a find_delta(f, x0, area_objetivo)
    RETORNAR delta, intervalo_E0 = (x0 - delta, x0 + delta)

    # Paso 2: Definir intervalo m s grande E0
    d0 = delta
    E0 = (x0 - d0, x0 + d0)

    # Paso 3: Construir familia de intervalos Ei con la f rmula
    intervalos = [E0]
    PARA i DESDE 1 HASTA n:
        d_i = d0 * (1 - 1 / (10^i))
        Ei = (x0 - d_i, x0 + d_i)
        AGREGAR Ei A intervalos
```

```
    FIN PARA

    # Paso 4: Verificar si x0 pertenece a todos los intervalos
    pertenece = VERIFICAR si x0 est  dentro de cada intervalo en intervalos

    # Paso 5: Imprimir resultados
        MOSTRAR d0
        MOSTRAR intervalo E0
        MOSTRAR cada intervalo Ei con su valor d_i
        MOSTRAR si x0 pertenece a todos los intervalos

    # Paso 6: Graficar resultados
        - Graficar la funci n f(x) en un rango alrededor de x0
        - Resaltar el  rea  bajo f(x) en E0
        - Dibujar l nea vertical en x0
        - Dibujar l mites de cada intervalo Ei
        - A adir etiquetas y leyenda

FIN FUNCI N
```

–0–

Pseudocode (in English) to define the kernel of the function $f_{Choice} := d_{\rightarrow}$

```
FUNCTION kernel_f_Choice(f, x0, target_area, n):

    # Step 1: Find delta (d0) so that the area under f(x) equals target_area
    CALL find_delta(f, x0, target_area)
    RETURN delta, interval_E0 = (x0 - delta, x0 + delta)

    # Step 2: Define largest interval E0
    d0 = delta
    E0 = (x0 - d0, x0 + d0)

    # Step 3: Build family of intervals Ei using formula
    intervals = [E0]
    FOR i FROM 1 TO n:
        d_i = d0 * (1 - 1 / (10^i))
        Ei = (x0 - d_i, x0 + d_i)
        ADD Ei TO intervals
    END FOR
```

```
    # Step 4: Check if x0 belongs to all intervals
    belongs = CHECK if x0 is inside every interval in intervals

    # Step 5: Print results
        PRINT d0
        PRINT interval E0
        PRINT each interval Ei with its d_i
        PRINT whether x0 belongs to all intervals

    # Step 6: Plot results
        - Plot function f(x) over a range around x0
        - Highlight area under f(x) in E0
        - Draw vertical line at x0
        - Draw boundaries of each interval Ei
        - Add labels and legend

END FUNCTION
```

$-0-$

Hence the kernel of the function $f_{Choice} := d_{\rightarrow}$ is

$$ker(d_{\rightarrow}) = \{x \in \{E\}_{i=1}^{\infty} | f_{Choice}(x) = e_{(Im\left(\int_{x-\delta}^{x+\delta} P_1(x)dx\right);\cdot)}\}$$

Now lets define the kernel of the function $f_{Cochoice} := d_{\leftarrow}$

Code in Python for to define the kernel of the function $f_{Cochoice} := d_{\leftarrow}$

Compiled code to define the kernel of the function $f_{Cochoice} := d_{\leftarrow}$ in matplotlib

$-0-$

```
import numpy as np
import matplotlib.pyplot as plt

# Funci n problema
def f(t):
    return t**2
```

```
# Intervalo del dominio
a, b = -2, 3
X_vals_global = np.linspace(a, b, 400)
Y_func = f(X_vals_global)

# Pendiente fija de las secantes
m_sec = 1

# Paso de x
step = 0.6

# Almacenar resultados
secantes_info = []
tangentes_info = []
intervalos = []

# Iterar desde el valor menor del dominio
x_values = np.arange(a, b, step)

for idx, x0 in enumerate(x_values, start=1):
    # Recta secante: y = m_sec*(t - x0) + f(x0)
    coef = [1, -m_sec, -(f(x0) - m_sec*x0)]
    roots = np.roots(coef)
    if len(roots) < 2:
        continue
    p1, p2 = np.sort(roots)

    # --- Punto central como el valor medio ---
    x = (p1 + p2) / 2
    y = f(x)

    # Distancias geom tricas SOLO en x
    d2 = abs(x - p1)
    d3 = abs(p2 - x)

    # Pendiente tangente m_paralell
    d0 = (p2 - p1)/2
    d1 = np.sqrt((p2 - p1)**2 + (f(p2) - f(p1))**2)
    theta = np.arccos(np.clip((2*d0)/d1, -1, 1)) if d1 != 0 else 0
    phi   = np.arccos(np.clip(d0/d2, -1, 1)) if d2 != 0 else 0
    omega = np.arccos(np.clip(d0/d3, -1, 1)) if d3 != 0 else 0

    d1d2 = theta - phi
    eta = np.deg2rad(180 - 90 - np.rad2deg(omega)) if d3 != 0 else 0
    d1d3 = np.deg2rad(180 - 90 - np.rad2deg(eta) - np.rad2deg(theta)) if d3 != 0
```

```
        m_paralell = ((y + d3*np.cos(d1d3)*np.sin(theta)) - (y - d2*np.cos(d1d2)*np.
                     ((x + d3*np.cos(d1d3)*np.cos(theta)) - (x - d2*np.cos(d1d2)*np.

        # -- Intervalo E_i --
        d_i = step * idx
        E_i = (x - d_i, x + d_i)
        intervalos.append((E_i, d_i))

        # Guardar info
        secantes_info.append(((p1, f(p1)), (p2, f(p2))))
        tangentes_info.append((x, y, m_paralell, d2, d3))

        if x0 >= p2:
            break

    # -- Gr fica --
    plt.figure(figsize=(10,7))
    plt.plot(X_vals_global, Y_func, label="f(x)=x^2", color="blue")

    # Dibujar secantes
    for (P1,P2) in secantes_info:
        p1, y1 = P1
        p2, y2 = P2
        b_line = f(p1) - m_sec*p1
        Y_sec = m_sec*X_vals_global + b_line
        plt.plot(X_vals_global, Y_sec, ':', label=f"Secante ({p1:.2f},{y1:.2f})-({p2
        plt.scatter([p1,p2],[y1,y2], color="orange", s=60, zorder=5)

    # Dibujar tangentes
    for (px,py,m,d2,d3) in tangentes_info:
        Y_tan = m*X_vals_global + (py - m*px)
        plt.plot(X_vals_global, Y_tan, '--', label=f"Tangente en ({px:.2f},{py:.2f})
        plt.scatter([px],[py], color="red", s=80, zorder=6)

    plt.title("Par bola, secantes y tangentes (intervalos E_i con d_i)")
    plt.legend()
    plt.grid(True)
    plt.show()

    # -- Resultados --
    for (P1,P2), (px,py,m,d2,d3), (E_i,d_i) in zip(secantes_info, tangentes_info, in
        p1,y1 = P1
        p2,y2 = P2
        print(f"Secante entre ({p1:.3f},{y1:.3f}) y ({p2:.3f},{y2:.3f})")
        print(f"   Punto central de tangencia: ({px:.3f},{py:.3f})")
```

```
    print(f"   Distancia en x al primer punto: {d2:.3f}")
    print(f"   Distancia en x al segundo punto: {d3:.3f}")
    print(f"   Intervalo E_i: ({E_i[0]:.3f}, {E_i[1]:.3f}), con d_i = {d_i:.3f}"

# ── Verificaci n: el punto central est  en todos los intervalos ──
central_points = [px for (px,py,m,d2,d3) in tangentes_info]
check_all = all(all(E[0] <= px <= E[1] for (E,d_i) in intervalos) for px in cent

print("\n El punto central est  en todos los intervalos?:", check_all)
```

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```
Secante entre (-2.000,4.000) y (3.000,9.000)
   Punto central de tangencia: (0.500,0.250)
   Distancia en x al primer punto: 2.500
   Distancia en x al segundo punto: 2.500
   Intervalo E_i: (-0.100, 1.100), con d_i = 0.600
Secante entre (-1.400,1.960) y (2.400,5.760)
   Punto central de tangencia: (0.500,0.250)
   Distancia en x al primer punto: 1.900
   Distancia en x al segundo punto: 1.900
   Intervalo E_i: (-0.700, 1.700), con d_i = 1.200
Secante entre (-0.800,0.640) y (1.800,3.240)
   Punto central de tangencia: (0.500,0.250)
   Distancia en x al primer punto: 1.300
   Distancia en x al segundo punto: 1.300
   Intervalo E_i: (-1.300, 2.300), con d_i = 1.800
Secante entre (-0.200,0.040) y (1.200,1.440)
   Punto central de tangencia: (0.500,0.250)
   Distancia en x al primer punto: 0.700
   Distancia en x al segundo punto: 0.700
   Intervalo E_i: (-1.900, 2.900), con d_i = 2.400
Secante entre (0.400,0.160) y (0.600,0.360)
   Punto central de tangencia: (0.500,0.250)
   Distancia en x al primer punto: 0.100
   Distancia en x al segundo punto: 0.100
   Intervalo E_i: (-2.500, 3.500), con d_i = 3.000
Secante entre (-0.000,0.000) y (1.000,1.000)
   Punto central de tangencia: (0.500,0.250)
   Distancia en x al primer punto: 0.500
   Distancia en x al segundo punto: 0.500
   Intervalo E_i: (-3.100, 4.100), con d_i = 3.600
```

El punto central est en todos los intervalos?: True

[Execution Finished]

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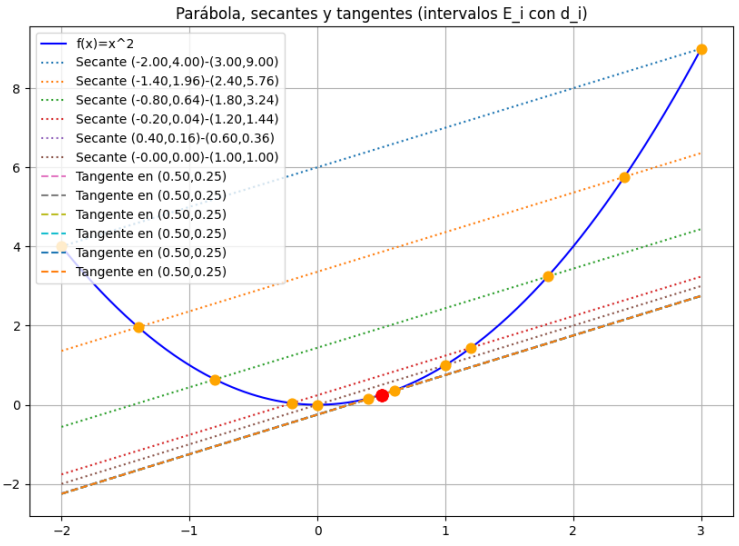


Hence

$$ker(d_{\leftarrow}) = \{x \in \{E\}_{i=1}^{\infty} | f_{Cochoice}(x) = e_{(Im\left(L_{tan(x,f(x))}\right);\cdot)}\}$$

–0–

Now lets define the image of $d_{\rightarrow}$ and $d_{\leftarrow}$

With

$$d_{\rightarrow} : (\{E_i\}_{i=0}^{\infty}; +) \mapsto (Im(\int_{x-\delta}^{x+\delta} P_1(x)dx); \cdot) \quad ;$$

$$d_{\leftarrow} : (\{E_i\}_{i=1}^{\infty}; +) \mapsto (Im\big(L_{tan(x,f(x))}\big); \cdot)$$

the map of the functions considered between the groups considered; the image of $d_{\rightarrow}$ is

$Im(d_{\rightarrow}) := Im\big(\int_{x-\delta}^{x+\delta} P_1(x)dx\big) \subseteq \Re$

i.e. the subset $Im(d_{\rightarrow})$ is defined by the interval of integration

while for $d_{\leftarrow}$

$Im(d_{\leftarrow}) := Im\big(L_{tan(x,f(x))}\big) \subseteq \Re$

i.e. the subset $Im(d_{\leftarrow})$ is defined by the values of $y := f(x)$ that define the tangent line L_{tan}

Finally lets define the quotient group

* Im $(d_{\rightarrow})$ is normal in $ker(d_{\rightarrow})$

Lets take $x_{Im} \in Im(d_{\rightarrow})$ such that $x :=$ area under the curve $f(x)$ between the interval $d_i = \delta_0(\#steep(i))$ and consider $x_{ker} \in ker(d_{\rightarrow})$

such that

$$x_{ker} \cdot x_{Im} \in x_{ker} \cdot Im(d_{\rightarrow})$$

and

$$x_{Im} \cdot x_{ker} \in Im(d_{\rightarrow}) \cdot x_{ker}$$

Like the product in $\Re$ is commutative hence

$$x_{ker} \cdot x_{Im} = x_{Im} \cdot x_{ker}$$

Hence

$$x_{ker} \cdot Im(d_{\rightarrow}) = Im(d_{\rightarrow}) \cdot x_{ker}$$

else like

$ker(d_{\rightarrow})$:= the set of all point where $f(x)$ is defined i.e. the interval that defines the domain of $f(x)$

$Im(d_{\rightarrow})$:= the area under the curve $f(x)$; defining the interval that establish the area between the interval that defines the domain of $f(x)$

If

$$|Im(d_{\rightarrow})| \leq |ker(d_{\rightarrow})|$$

on the considered interval hence

$$Im(d_{\rightarrow}) \subseteq ker(d_{\rightarrow})$$

Hence

$Im(d_{\rightarrow})$ is normal in $ker(d_{\rightarrow})$

* Quotient Group

Hence the product of two lateral classes is

$(x_{ker} \cdot Im(d_{\rightarrow}))(x_{ker} \cdot Im(d_{\rightarrow})) = (x_{ker} \cdot x_{ker} Im(d_{\rightarrow}))$

Hence

$$ker(d_{\rightarrow})/Im(d_{\rightarrow}) = \{x_{ker} \cdot Im(d_{\rightarrow}) | x_{ker} \in ker(d_{\rightarrow})\}$$

is the quotient group

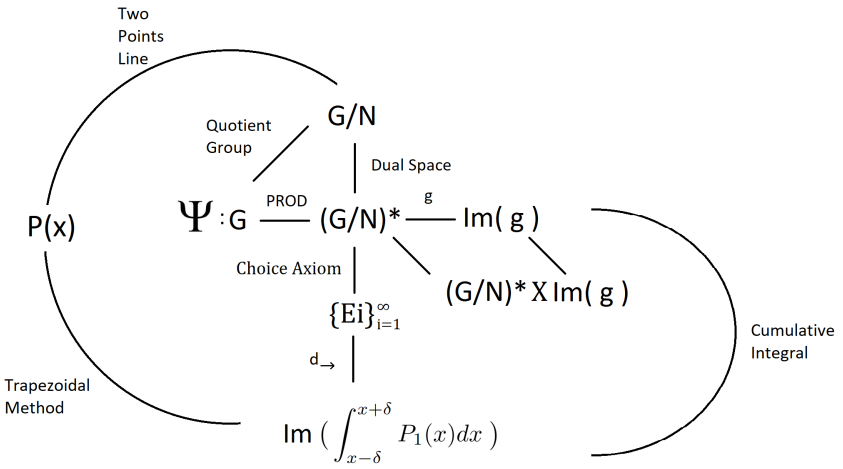
* Homology Group

Hence

$$H_{d_{\rightarrow}} = \frac{ker(d_{\rightarrow})}{im(d_{\rightarrow})}$$

is the "Homology group"

The relation between G and G' is given by next diagram (before the chain $\cdots \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} A^3 \xrightarrow{d^3} A^4 \xrightarrow{d^4} \cdots$)



with

$$G' := (G/N)' \times Im(g)$$

where G and A are vector spaces

**** Im $(d_{\leftarrow})$ is normal in $ker(d_{\leftarrow})$ ****

Lets take $x_{Im} \in Im(d_{\leftarrow})$ such that $x :=$ height of the curve in the point x on the interval $d_i = \delta_0(\#steep(i))$ and consider $x_{ker} \in ker(d_{\leftarrow})$

such that

$$x_{ker} \cdot x_{Im} \in x_{ker} \cdot Im(d_{\rightarrow})$$

and

$$x_{Im} \cdot x_{ker} \in Im(d_{\rightarrow}) \cdot x_{ker}$$

Like the product in $\Re$ is commutative hence

$$x_{ker} \cdot x_{Im} = x_{Im} \cdot x_{ker}$$

Hence

$$x_{ker} \cdot Im(d_{\leftarrow}) = Im(d_{\leftarrow}) \cdot x_{ker}$$

else like

$ker(d_{\leftarrow})$:= the set of all points x where $f_{Cochoice}(x) := y = xm_{\parallel} + b = e_{Im(L_{tan(x,f(x))})}$ is defined

$Im(d_{\leftarrow})$:= the set of all points $y = m_{\parallel}x + b$

If

$$|Im(d_{\leftarrow})| \leq |ker(d_{\leftarrow})|$$

on such intervals considered hence

$$Im(d_{\leftarrow}) \subseteq ker(d_{\leftarrow})$$

Hence

$Im(d_{\leftarrow})$ is normal in $ker(d_{\leftarrow})$

* Quotient Group

Hence the product of two lateral classes is

$(x_{ker} \cdot Im(d_{\leftarrow}))(x_{ker} \cdot Im(d_{\leftarrow})) = (x_{ker} \cdot x_{ker} Im(d_{\leftarrow}))$

Hence

$$ker(d_{\leftarrow})/Im(d_{\leftarrow}) = \{x_{ker} \cdot Im(d_{\leftarrow}) | x_{ker} \in ker(d_{\leftarrow})\}$$

is the quotient group

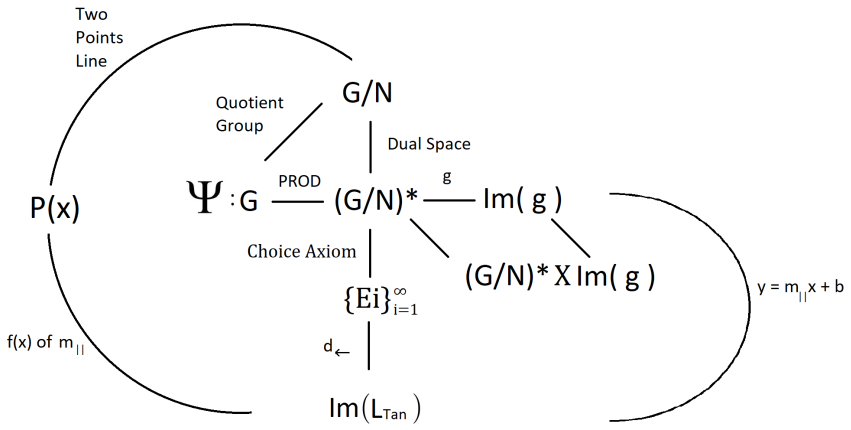
* Cohomology Group

Hence

$$H_{d_{\leftarrow}} = \frac{ker(d_{\leftarrow})}{im(d_{\leftarrow})}$$

is the "Cohomology group"

The relation between G and G' is given by next diagram (before the chain $\cdots \xleftarrow{d_0} A_0 \xleftarrow{d_1} A_1 \xleftarrow{d_2} A_2 \xleftarrow{d_3} A_3 \xleftarrow{d_4} A_4 \xleftarrow{d_5} \cdots$)



with

$$G' := (G/N)' \times Im(g)$$

where G and A are vector spaces

*

6 Riemann manifold is a Kähler manifold

A Kähler manifold is a Riemannian manifold X of even dimension 2n whose holonomy group is contained in the unitary group $U(n)$. Equivalently, there is a complex structure J on the tangent space of X at each point (that is, a real linear map from TX to itself $J^2 = -1$) such that J preserves the metric g (meaning that $g(Ju, Jv) = g(u, v)$ and J is preserved by parallel transport.

Def. A smooth manifold M endowed with a Riemannian metric g is a Riemannian manifold, denoted (M, g)

Def. Let E be a rank-k vector bundle over a smooth manifold M, and let ∇ be a connection on E. Given a piecewise smooth loop $\gamma : [0, 1] \to M$ based at x in M, the connection defines a parallel transport map $P_\gamma : E_X \to E_X$ on the fiber of E at x. This map is both linear and invertible, and so defines an element of the general linear group $GL(E_X)$- The holonomy group of ∇ based at x is defined as

$$Hol_x(\nabla) = \{P_\gamma \in GL(E_x) | \gamma \text{ is a loop based at } x\}$$

Def (before). Let $\pi : E \to M$ be a smooth vector bundle over the manifold M. A **connection** (often called a **covariant derivative**) on E is a map

$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ that assigns to each smooth section $s \in \Gamma(E)$ a new section ∇s, which is a 1-form on M with values in E.

It must satisfy two properties:

1. **Linearity over** R**:**$\nabla(s_1 + s_2) = \nabla s_1 + \nabla s_2$.
2. **Leibniz rule with respect to functions:** For $f \in C^\infty(M)$ and $s \in \Gamma(E)$

$\nabla(fs) = df \otimes s + f\nabla s$

where $\Gamma(E)$ are the Christoffel symbols

Def (Now). Let $\pi : E \to X$ be the fiber over the base space X. A **connection** (often called a **covariant derivative**) on E is a map

$\nabla : E \to T^*M \otimes E$ that assigns to each smooth section $s : CSB_{(h_1,h_2)} \to E$ a new section ∇s, which is a 1-form on M with values in E.

It must satisfy two properties:

Linearity over R**:**$\nabla(s_1 + s_2) = \nabla s_1 + \nabla s_2$.

Def. Given a Riemannian manifold (M, g), the Christoffel symbols Γ^k_{ij} are:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\delta_i g_{jl} + \delta_j g_{il} - \delta_l g_{ij})$$

where:

- g_{ij} are the components of the metric tensor,
- g^{kl} is the inverse of the metric matrix,
- δ_i denotes partial differentiation with respect to the coordinate x^i.

Def. A piecewise function is a function whose domain is partitioned into several intervals on which the function may be defined differently.

Def. Let X be a topological space. A loop is a continuous function $f : [0, 1] \to X$ such that $f(0) = f(1)$. If f begins and ends at $x_0 \in X$ the loop is said to be based at x_0. A loop is then a path that begins and ends at the same point x_0.

The set of homotopy classes of loops based at x_0 together with the operation of path composition, forms the fundamental group of X relative to x_0, usually denoted by $\pi_1(X, x_0)$.

Def. Parallel transport of tangent vectors is a special case of a more general construction involving an arbitrary vector bundle E. Specifically, parallel transport of tangent vectors is the case where E is the tangent bundle TM.

Let M be a smooth manifold. Let $E \to M$ be a vector bundle with connection ∇ and $\gamma : I \to M$ a curve parameterized by an open interval I. A section X of E along γ is called parallel if

$$\nabla_{\dot{\gamma}(t)} X = 0 \text{ for } t \in I$$

where

$\dot{\gamma}(t) :=$ tangent vector

Def. If V is a vector space over the field F, the general linear group of V, written $\mathrm{GL}(V)$ or $\mathrm{Aut}(V)$, is the group of all automorphisms of V, i.e. the set of

all bijective linear transformations $V \to V$, together with functional composition as group operation.

Def. In mathematics, the unitary group of degree n, denoted $U(n)$, is the group of $n \times n$ unitary matrices, with the group operation of matrix multiplication.

The unitary group $U(n)$ is a real Lie group of dimension n^2. The Lie algebra of $U(n)$ consists of $n \times n$ skew-Hermitian matrices, with the Lie bracket given by the commutator.

Proof.

Like $\mathbb{C}$ is a manifold (symplectic and complex) equipped with g the riemannian metric hence $(\mathbb{C}, g)$ is Riemannian manifold.

Now lets define the connection on the rank-k vector bundle over the manifold M

Remember that a complex vector bundle consist of:

1. A topological space $X := B \subseteq \mathbb{C}$ (base space) and $E := \mathbb{C}$ (total space)

$B = \{(x_1', x_2') \in \mathbb{C} \quad | \quad (x_1', x_2') \in \bigcup_{i \in Z^+ \cup Z^-} S_p M_i \quad \wedge \quad (x_1', x_2') \in \bigcap_{i \in Z^+ \cup Z^-} S_p M_i\}$

2. A continuous surjection $\pi : E \to X$

3. For every $x \in X \quad [\quad x := (x_1, x_2) \quad]$, the structure of a finite-dimensional complex vector space on the fiber $\pi^{-1}(\{x\})$

$\pi = \{(x_1, x_2) \in \mathbb{C} \quad | \quad (x_1, x_2) \in C \wedge (x_1', x_2') \in S_p M \wedge (x_1', x_2') \in B \wedge (x_1 \neq x_1' \quad ; \quad x_2 \neq x_2')\}$ of rank 2 by $\pi(x_1, x_2) = (x_1', x_2')$

Now lets consider the Christoffel symbol for the Riemannian metric ($g_p := g_{12}$)

The Christoffel symbol for $g_p :=< (x_1, f(x_1)); (x_2, f(x_2)) >= detA$

is

$$g_p := g_{jl} = g_{il} = g_{ij} = detA$$

and

$$g^{kl} = \frac{1}{detA}$$

Hence

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\delta_i g_{jl} + \delta_j g_{il} - \delta_l g_{ij})$$

$$\Gamma^i_{ii} = \frac{1}{2} g^{ii}(\delta_i g_{ii} + \delta_i g_{ii} - \delta_i g_{ii})$$

$$= \frac{1}{2} g^{ii}(\delta_i g_{ii})$$

$$= \frac{1}{2} \frac{1}{detA}(\delta_i g_{ii})$$

Remember that: " In a system of linear equations where A is the matrix of coefficients; if A is a $2x2$ matrix and the $detA = 0$; the lines that define A are parallel "

Hence the Christoffel symbol is non defined by det A = 0

Now lets consider the smooth section

$$\sigma : CSB_{(h_1,h_2)} \to E$$

Where the point (x_1^*, x_2^*) is a cosecant point of the line L while the set

$$CSB_{(0,0)} = \{(x_1^*, x_2^*) \in L_{\text{secant}} \quad |$$

1) $(x_a, x_b) \in B_r(x_a, x_b)$i.e. is the center of such circle ;

$$2)(x_1, x_2) \in B_r(x_a, x_b) \quad ;$$

$$3)\sqrt{(x_a - x_1)^2 + (x_b - x_2)^2} = r \quad ;$$

$$4)\sqrt{(x_a - x_1^*)^2 + (x_b - x_2^*)^2} = p \quad ;$$

$$5) r^2 + p^2 = q^2 \quad ;$$

$$6)\sqrt{(x_1^* - x_1)^2 + (x_2^* - x_2)^2} = q \quad ;$$

such that $\forall(x_1, x_2)$ and $(x_1', x_2') \in L_{secant} \subset C_{(0,0)}; (x_1^*, x_2^*)$ is a cosecant point of $\mathrm{L}_{secant}\}$

defining the set of cosecant points for the circle $C_{(0,0)}$.

Later by translation $=_T$ of the point (x_a, x_b) center of the circle C

$CSB_{(h_1,h_2)} = \{(x_1^*, x_2^*) \in L_{secant} \quad | \quad 1), 2), 3), 4), 5)$ and $5)$ are fulfilled such that

$\forall(x_1, x_2)$ and $(x_1', x_2') \in C_{(h_1,h_2)}; (x_1^*, x_2^*)$ is cosecant point of $L_{\mathrm{secant}_{[(x_1,x_2),(x_1',x_2')]}}\}$

i.e. such set is the set of all cosecant points on $\mathbb{C}$

Like such line L and cosecant point are defined for all $\Re^2$

Hence for every h_1, h_2, x_1 and x_2 defining an open set on $\mathbb{C}$;

$CSB_{(h_1,h_2)} = \mathbb{C}$ ($\Re^2$ endowed with $-a =_A a$ to have the properties of $\mathbb{C}$)

Now lets consider next statement about "Connection " [see Connection (Vector Bundle)]

" *In mathematics, and especially differential geometry and gauge theory, a connection on a fiber bundle is a device that defines a notion of parallel transport on the bundle; that is, a way to "connect" or identify fibers over nearby points. The most common case is that of a linear connection on a vector bundle, for which the notion of parallel transport must be linear. A linear connection is*

equivalently specified by a covariant derivative, an operator that differentiates sections of the bundle along tangent directions in the base manifold, in such a way that parallel sections have derivative zero. "

This means that to define a connection is necessary to differentiate two sections of a fiber bundle.

The connection definition considers the "operator" $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E^*)$

Lets construct now the set $T^*M \otimes E^*$

with

$$(x_1^*, x_2^*) \in CSB_{(h_1,h_2)}$$

and

$$\sigma(x_1^*, x_2^*) \in E$$

Case 1)

Lets take $L_{\text{secant}_{[(x_1,x_2),(x_1',x_2')]}}$ (Note that L_{secant} definition is the same if the circle is out of the origin i.e. $C_{(h_1,h_2)}$)

$$L_{\text{secant}_{[(x_1,x_2),(x_1',x_2')]}} = \{(x_1^*, x_2^*) \in C_{(h_1,h_2)} \subset E \quad | \quad x_2^* = (\frac{x_2' - x_2}{x_1' - x_1})(x_1^* - x_1) + x_2 \quad ;$$

$$d = \sqrt{(x_1' - x_1)^2 + (x_2' - x_2)^2} \quad ;$$

$$(x_1, x_2) \in B \quad \text{and} \quad (x_1', x_2') \in SB\}$$

And lets take the covector of $L_{\text{secant}_{[(x_1,x_2),(x_1',x_2')]}}$

with

$$d[(x_1, x_2), (x_1^*, x_2^*)] = q(t)_{E/N_E} \in (E/N_E)^*$$

where

$(E/N_E)^*$:= Dual space of the quotient group of E

with $x_2 = f(x_1)$ and $x_2^* = f(x_1^*)$

Now lets define the interval $[x_1, x_2]$ and the Choice Axiom for Intersection hence

$x_1 = x - \delta$

$x_2 = x + \delta$

And lets consider the derivative of $f(x)$ on the point $(x, f(x)) \in T_{p_T}M$

Remember that

$T_{p_T}M = \{ \quad p_T := (x_1^*, f(x_1^*)) \in \mathbb{C} \quad |$

$$p := (x_1, x_2) \in S_pM \quad ;$$

$$[x_1 = x_1^* - \delta] < x_1^* < x_1^* + \delta \quad ;$$

$$x_2 = f(x_1^* - \delta) < f(x_1^*) < f(x_1^* + \delta)$$

(or

$$x_1^* - \delta < x_1^* < (x_1 = x_1^* + \delta) \quad ;$$

$$f(x_1^* - \delta) < f(x_1^*) < [x_2 = f(x_1^* + \delta)] \qquad) \quad ;$$

$$m_{\parallel} = \frac{(f(x_1^*) + d_3 cos(\angle d_1 d_3) sin(\theta)) - (f(x_1^*) - d_2 cos(\angle d_1 d_2) sin(\theta))}{(x_1^* + d_3 cos(\angle d_1 d_3) cos(\theta)) - (x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta))} \quad ;$$

$$y = m_{||}(x - x_1^*) + f(x_1^*) \quad ;$$

$$(x, y) \in \mathbb{C}$$

$$\}$$

With $(x_1^* - \delta, f(x_1^* - \delta))$ and $(x_1^* + \delta, f(x_1^* + \delta))$ the to points that defines the parallel line its coordinates are

$$x_{\|\delta+} = d_3 cos(\angle d_1 d_3) cos(\theta) + x_1^*$$

$$y_{\|\delta+} = d_3 cos(\angle d_1 d_3) sin(\theta) + f(x_1^*)$$

$$x_{\|\delta-} = x_1^* - d_2 cos(\angle d_1 d_2) cos(\theta)$$

$$y_{\|\delta-} = f(x_1^*) - d_2 cos(\angle d_1 d_2) sin(\theta)$$

Hence the covector of L_{tan} is

$$d[(x_{\|\delta-}; y_{\|\delta-}); (x_{\|\delta+}); (y_{\|\delta+})] = q(t)_{T_{p_T} M / N_{T_{p_T} M}} \in (T_{p_T} M / N_{T_{p_T} M})^*$$

Considering its tensor product

$$q(t)_{T_{p_T} M / N_{T_{p_T} M}} \otimes q(t)_{E/N_E} = \left\{ \begin{array}{c} (q(t)_{T_{p_T} M / N_{T_{p_T} M}}, q(t)_{E/N_E}) \in (T_{p_T} M / N_{T_{p_T} M})^* \times (E/N_E)^* \\ q(t)_{T_{p_T} M / N_{T_{p_T} M}} e_1 + q(t)_{E/N_E} e_2 = c \in R = T^2(x) \end{array} \right.$$

$$= detA =< (q(t)_{T_{p_T} M / N_{T_{p_T} M}}, q(t)_{E/N_E}), (e_1, e_2) >$$

**

With $(x_1^*, x_2^*) \in CSB_{(h_1, h_2)}$ and $\sigma(x_1^*, x_2^*) \in E$ since $\sigma : CSB_{(h_1, h_2)} \to E$

For two sections

$(x_1^*, x_2^*) \in CSB_{(h_1,h_2)_1}$ and $(x_a^*, x_b^*) \in CSB_{(h_1,h_2)_2}$

If

$L_{Sec1} = q(t)_{E/N_E}$ and $L_{Tan1} = q(t)_{T_{p_T}M/N_{T_{p_T}M}}$

are equal to

$L_{Sec2} = q(t)_{E/N_E}$ and $L_{Tan2} = q(t)_{T_{p_T}M/N_{T_{p_T}M}}$

(i.e. $L_{sec1} = L_{sec2}$ and $L_{tan1} = L_{tan2}$)

Hence section 1 is equal to section 2 ($\sigma_1 = \sigma_2$)

Case 2)

Now lets take

$$(x_1^*, x_2^*) \in CSB_{(h_1,h_2)_1}$$

and

$$(x_a^*, x_b^*) \in CSB_{(h_1,h_2)_2}$$

and lets apply $Prod_{map}$ function

$$Prod(x_1^*, x_2^*) = x_1^* \cdot x_2^* = q(t)_{E/N_E(12)}$$

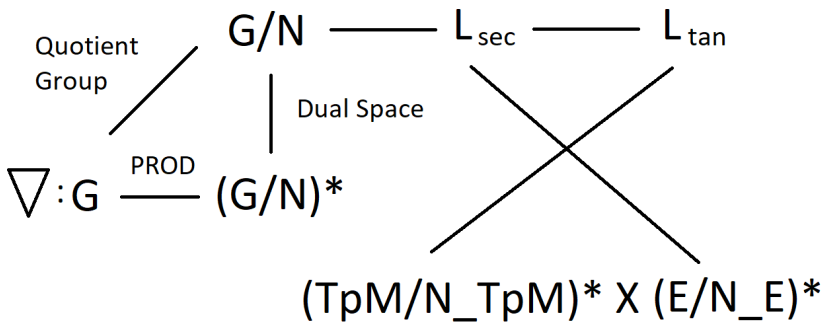
$$Prod(x_a^*, x_b^*) = x_a^* \cdot x_b^* = q(t)_{E/N_E(ab)}$$

If

$$q(t)_{E/N_E(12)} = q(t)_{E/N_E(ab)}$$

hence

$$\sigma_1 = \sigma_2$$



** Linearity over R:

Lets consider

$$\nabla(CSB_1 \cup CSB_2) \to E' \cup E'' \to (E' \cup E'')/(N'_E \cup N''_E)$$

with

$$E' \subseteq E$$

and

$$E'' \subseteq E$$

Hence

$$(T_{p_T} M' \cup T_{p_T} M''/(N_{T_{p_T} M'} \cup N_{T_{p_T} M''}))^* \times (E' \cup E''/(N'_E \cup N''_E))^*$$

And now lets consider

$$\nabla CSB_1 \to E' \to E'/N'_E$$

$$\nabla CSB_2 \to E'' \to E''/N''_E$$

Hence

$$(T_{p_T} M'/N_{T_{p_T} M'})^* \times (E'/N'_E)^*$$

$$(T_{p_T} M''/N_{T_{p_T} M''})^* \times (E''/N''_E)^*$$

If

$$(T_{p_T} M' \cup T_{p_T} M''/(N_{T_{p_T} M'} \cup N_{T_{p_T} M''}))^* \times (E' \cup E''/(N'_E \cup N''_E))^*$$

$$=$$

$$(T_{p_T} M'/N_{T_{p_T} M'})^* \times (E'/N'_E)^* \cup (T_{p_T} M''/N_{T_{p_T} M''})^* \times (E''/N''_E)^*$$

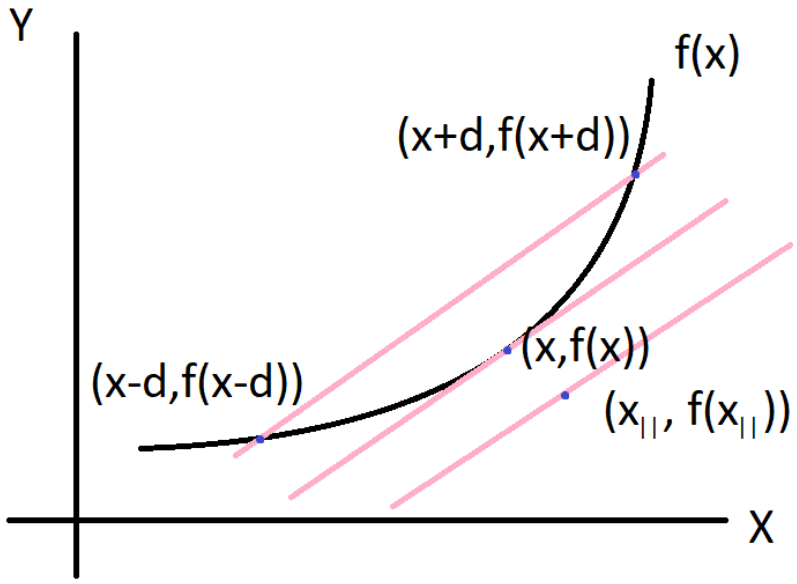
Hence

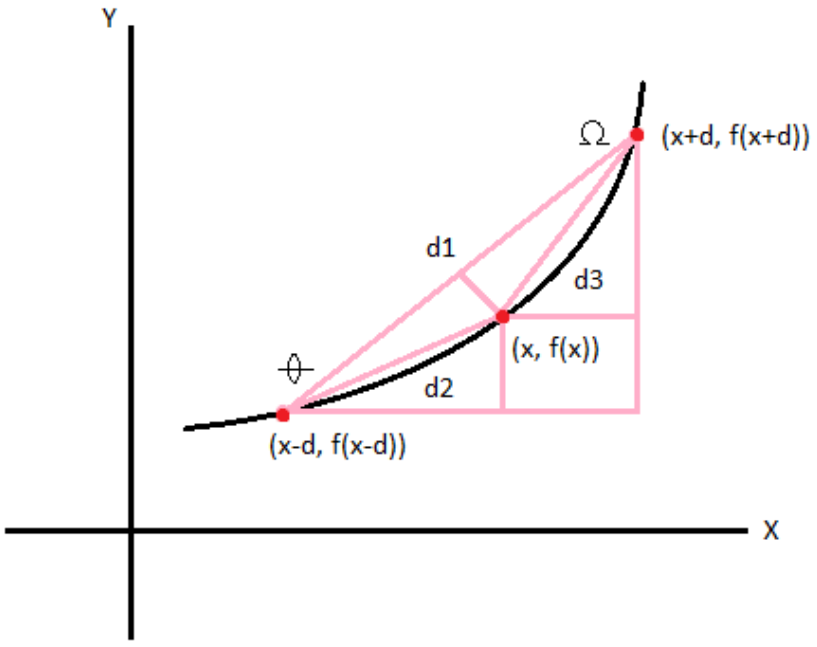
$$\nabla(CSB_1 \cup CSB_2) = \nabla(CSB_1) \cup \nabla(CSB_2)$$

If CSB_1 and CSB_2 intersect or are disjoint

* Parallel Transport

Lets establish next definition about " Parallel Transport "


Y
f(x)
(x+d,f(x+d))
(x,f(x))
(x-d,f(x-d))
(x|| , f(x||))
X


Y
Ω
(x+d, f(x+d))
d1
d3
(x, f(x))
d2
(x-d, f(x-d))
X

Lets take the triangle defined by d_1, d_2, d_3 segments and lets apply cosine theorem

$$d_2^2 = d_1^2 + d_3^2 - 2d_1 d_3 cos(\Omega)$$

$$d_3^2 = d_1^2 + d_2^2 - 2d_1 d_2 cos(\theta)$$

Hence

$$\Omega = arccos(\frac{d_1^2 + d_3^2 - d_2^2}{2d_1 d_3})$$

$$\theta = arccos(\frac{d_1^2 + d_2^2 - d_3^2}{2d_1 d_2})$$

Now lets obtain the height of the triangle h

$$x_{d_3} = d_3 sin(\Omega) \quad ;$$

$$x_{d_2} = d_2 sin(\theta)$$

Hence

$$d_1 = x_{d_3} + x_{d_2}$$

The equation that pass through two points is

$$m = \frac{f(x+\delta) - f(x-\delta)}{(x+\delta) - (x-\delta)}$$

$$y - f(x-\delta) = m(x - (x-\delta))$$

distance to the point of intersection with height h is

$$d_{-\delta;h} = \sqrt{(x_h - (x-\delta))^2 + (y_h - f(x-\delta))^2}$$

with

$$y_h = m(x_h - (x-\delta)) + f(x-\delta)$$

and

$$d_{(-\delta;h)} = x_{d_2}$$

$$d_{-\delta;h} = \sqrt{(x_h - (x-\delta))^2 + (m(x_h - (x-\delta)) + f(x-\delta) - f(x-\delta))^2}$$

$$d^2_{(-\delta,h)} = x_h^2 - 2x_h(x-\delta) + (x-\delta)^2 + m^2(x_h - (x-\delta))^2$$

Last term of equation is (this clearing was made by Copilot Microsoft Bing https://copilot.microsoft.com/shares/amf3S66dq7FVDn1R7tC67)

$$m^2(x_h - (x-\delta))^2 = m^2(x_h^2 - 2x_h(x-\delta) + (x-\delta)^2)$$

Hence

$$d^2 = x_h^2 - 2x_h(x-\delta) + (x-\delta)^2 + m^2x_h^2 - 2m^2x_h(x-\delta) + m^2(x-\delta)^2$$

Simplifying

$$d^2 = (1+m^2)x_h^2 - 2(1+m^2)x_h(x-\delta) + (1+m^2)(x-\delta)^2$$

In quadratic form

$$(1+m^2)x_h^2 - 2(1+m^2)(x-\delta)x_h + (1+m^2)(x-\delta)^2 - d^2 = 0$$

Let:

- $(A = 1 + m^2)$
- $(B = -2(1 + m^2)(x - \delta))$
- $(C = (1 + m^2)(x - \delta)^2 - d^2)$

Hence $x_h = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

Simplifying

$x_h = \frac{2(1+m^2)(x-\delta) \pm \sqrt{[2(1+m^2)(x-\delta)]^2 - 4(1+m^2)((1+m^2)(x-\delta)^2 - d^2)}}{2(1+m^2)}$

Simplifying root

$$= 4(1 + m^2)^2(x - \delta)^2 - 4(1 + m^2)\big((1 + m^2)(x - \delta)^2 - d^2\big)$$

$$= 4(1 + m^2)d^2$$

Hence

$$x_h = (x - \delta) \pm \frac{d}{\sqrt{1 + m^2}}$$

and

$$y_h = m((x - \delta) \pm \frac{d_{(-\delta,h)}}{\sqrt{1 + m^2}} - (x - \delta)) + f(x - \delta)$$

Besides

$$h = d_3 cos(\Omega) \quad ;$$

$$h = d_2 cos(\theta)$$

Now lets obtain the straight line that pass through two points

$$m_2 = \frac{f(x) - y_h}{x - x_h}$$

with $y := y_{2h}$

$$y_{2h} - y_h = m_2(x_{2h} - x_h)$$

$$d_{(h;2h)} = \sqrt{(x_{2h} - x_h)^2 + (y_{2h} - y_h)^2}$$

Substituting

$$d_{(h;2h)} = \sqrt{(x_{2h} - x_h)^2 + (m_2(x_{2h} - x_h) + y_h - y_h)^2}$$

Lets clear x_{2h}

$$d_{(h;2h)} = \sqrt{(x_{2h} - x_h)^2 + \big(m_2(x_{2h} - x_h)\big)^2}$$

$$= \sqrt{(x_{2h} - x_h)^2\big(1 + m_2^2\big)}$$

$$d_{(h;2h)} = |x_{2h} - x_h|\sqrt{1 + m_2^2}$$

$$|x_{2h} - x_h| = \frac{d_{(h;2h)}}{\sqrt{1 + m_2^2}}$$

Hence the value of $x = x_{2h}$

$$x_{2h} - x_h = \pm\frac{d_{(h;2h)}}{\sqrt{1 + m_2^2}}$$

$$x_{2h} = x_h \pm \frac{d_{(h;2h)}}{\sqrt{1 + m_2^2}}$$

Hence the value of $y = y_{2h}$

$$y_{2h} = y_h + m_2(x_h \pm \frac{d_{(h;2h)}}{\sqrt{1+m_2^2}} - x_h)$$

$$y_{2h} = y_h + m_2(x_h \pm \frac{d_{(h;2h)}}{\sqrt{1+m_2^2}})$$

Hence the parallel function is defined for each point with next conditions

If

$f(x) > y_{2h}$ this coordinate is subtracted

$x < x_{2h}$ this coordinate is added

$f(x) < y_{2h}$ this coordinates is added

$x > x_{2h}$ this coordinate is subtracted

$x = x_{2h}$ this coordinate maintain

$y = y_{2h}$ this coordinate maintain

Now lets define the parallel line to L_{tan} that pass through $(x_{2h}, f_{\parallel}(x_{2h}))$

with

$f_{\parallel}$:= the parallel function

Lets establish the points where L_{tan} intersect $f_{\parallel}$ defined like

$$m_{tan} = \frac{y_{+\parallel} - y_{\parallel-\delta}}{x_{\parallel+\delta} - x_{\parallel-\delta}}$$

$$y_{tan} - y_{\parallel-\delta} = m_{tan}(x_{tan} - x_{\parallel-\delta})$$

with

$$f_{\parallel}(x) = y_{tan}$$

and

$$f_{\parallel}(x = x_{tant})$$

the values of both points can be determined

Lets called such points $x_{tan-\delta}$, $x_{tan+\delta}$, $y_{tan-\delta}$ y $y_{tan+\delta}$ such that

$$x_{tan-\delta} < x_{2h} < x_{tan+\delta} \quad ;$$

$$y_{tan-\delta} < y_{2h} < y_{tan+\delta}$$

Lest obtain

$$\delta_2 := x_{2h} - x_{tan-\delta} \quad ;$$

$$\delta_2 := x_{tan+\delta} - x_{2h}$$

By Axiom Choice for Intersection x_{2h} is the point where all segments defined by $\delta_i = d_2(\#steep(i))$ intersect

With this points lets define d_1, d_2, d_3, Ω and θ to obtain $m_{\parallel}$ for the line that pass through $(x_{2h}, f(x_{2h}))$ hence

$$x_{f_{\parallel}+\delta} = d_3 cos(\angle d_1 d_3) cos(\theta) + x_{2h}$$

$$y_{f_{\parallel}+\delta} = d_3 cos(\angle d_1 d_3) sin(\theta) + f_{\parallel}(x_{2h})$$

$$x_{f_{\parallel}-\delta} = x_{2h} - d_2 cos(\angle d_1 d_2) cos(\theta)$$

$$y_{f_{\parallel}-\delta} = f_{\parallel}(x_{2h}) - d_2 cos(\angle d_1 d_2) sin(\theta)$$

Hence

$$d[(x_{tan-\delta}, x_{tan+\delta}); (y_{tan-\delta}, y_{tan+\delta})] = q(t)_{T_{p_{f_\parallel}} M/N_{T_{p_{f_\parallel}} M}}$$

$$d[(x_{f_\parallel-\delta}, x_{f_\parallel+\delta}); (y_{f_\parallel-\delta}, y_{f_\parallel+\delta})] = q(t)_{E_{f_\parallel}/N_{E_{f_\parallel}}}$$

Hence

$$q(t)_{E_{f_\parallel}/N_{f_\parallel}} \otimes q(t)_{T_{p_{f_\parallel}} M/N_{T_{p_{f_\parallel}} M}} = \left\{ \begin{array}{c} (q(t)_{E_{f_\parallel}/N_{f_\parallel}}, q(t)_{T_{p_{f_\parallel}} M/N_{T_{p_{f_\parallel}} M}}) \in (E_{f_\parallel}/N_{f_\parallel})^* \times (T_{p_{f_\parallel}} M/N_{T_{p_{f_\parallel}} M})^* \\ q(t)_{E_{f_\parallel}/N_{f_\parallel}} e_1 + q(t)_{T_{p_{f_\parallel}} M/N_{T_{p_{f_\parallel}} M}} e_2 = c \in R = T^2(X) \end{array} \right.$$

$$= detA =< (q(t)_{E_{f_\parallel}(N_{f_\parallel}}, q(t)_{T_{p_{f_\parallel}} M/N_{T_{p_{f_\parallel}} M}}); (e_1, e_2) >$$

Now lets establish the loop function γ with the points of the straight line $L_\perp$

With

$$y - y_h = m_2(x - x_h)$$

the perpendicular line to the secant line that pass through $[x - \delta, x + \delta]$ on the point parallel to $(x, f(x))$ i.e. $(x_h, f(x_h))$

Lets define the γ function with the points y_{sec}, y_{tan} and $y_{f_\parallel}$ such that $y_{sec} := y_1$; $y_{tan} := y_2$ and y_{tan} is on the secant of the parallel function $f_\parallel$ obtained such that

$$y \in Dom(\gamma) := [y_{sec} \quad ; y_{tan} \quad ; \quad y_{f_\parallel}; ...]$$

$$\gamma : Dom(\gamma) \to [0, 1]$$

Hence the function γ define the set

$$Dom(\gamma) = y_\gamma = \{ \quad y \in \Re \quad | \quad y = m_2(x - x_h) + y_h \quad ; \quad x = \gamma(y) \in [0,1] \quad \}$$

To fulfill the definition about a "loop function" i.e. $\gamma(0) = \gamma(1)$ hence γ must be a periodic function

* Examples of loop functions

* $f(x) = x(1-x)$

* $f(x) = x^2 - x$

* $f(x) = sin(2\pi x)$

If the function is partitioned into several domains hence is piecewise

* Example

$$\gamma := \begin{cases} cos(2\pi x) & ; x \in (0; 0.5) \\ cos(2\pi x) & ; x \in (0.5; 1) \end{cases}$$

Hence the function γ is defined for each y_i of $y_\perp$

Note that γ is not injective by periodicity

Now with the connection defined like $T_{p_T}M$

$$\nabla : T_{p_T}M \to (E_{f_\parallel}/N_{E_{f_\parallel}})^* \otimes (T_{p_{f_\parallel}}M/N_{T_{p_{f_\parallel}}M})^*$$

A section $\sigma_{f_\parallel} : CSB_{f_\parallel} \to E_{f_\parallel}$ is parallel if $f_\parallel(x)$ to $f(x)$ is parallel (see early conditions about parallel function)

Hence the connection ∇ defines a parallel transport map from $T_{p_T}M$ to $E_{f_\parallel}$ on the fiber of E at $(x_{f_\parallel}, f_\parallel(x_{f_\parallel}))$

* Invertible map

Lets take γ and the set $Im(\gamma)$ i.e. $x_i \in [0,1]$ and lets obtain the values of y to define the set y_γ

with

$$y_i = m_2(x_i - x_h) + y_h$$

Hence

$$y_\gamma = \{ y_i \in \Re \quad | \quad y_i = m_2(x_i - x_h) + y_h \quad ; \gamma(y) = x \in [0,1] \quad \}$$

Now lets consider the function $f(x)_\parallel$ and lets define the points

$$x_\parallel = f_\parallel^{-1}(f_\parallel(x))$$

Now lets define the perpendicular straight line to

$L_\perp := y_i = m_2(x_i - x_h) + y_h$

that intersect $f_\parallel(x)$ and defines a secant line

with $m_2' = \frac{1}{m_2}$

such straight line is

$$L_{sec} := y_i = m_2'(x_i - x_h) + y_h$$

with

$$q(t)_{E_{f_\parallel}/N_{f_\parallel}} \otimes q(t)_{T_{p_{f_\parallel}}M/N_{T_{p_{f_\parallel}}M}}$$

if

$$d[x_{\parallel-\delta}, x_{\parallel+\delta}] = q(t)_{T_{p_{f_\parallel}}M/N_{T_{p_{f_\parallel}}M}}$$

with

$$d[x_{\parallel-\delta}, x_{\parallel+\delta}] = \sqrt{(x_{\parallel+\delta} - x_{\parallel-\delta})^2 + (f_\parallel(x_{\parallel+\delta}) - f_\parallel(x_{\parallel-\delta}))^2}$$

and

$$f_{\|}(x_{\|+\delta}) = m_2'(x_{\|+\delta} - x_h) + y_h$$

$$f_{\|}(x_{\|-\delta}) = m_2'(x_{\|-\delta} - x_h) + y_h$$

with

$$x_{\|+\delta} = x_{\|} + \delta$$

$$x_{\|-\delta} = x_{\|} - \delta$$

the intersection points of L_{sec} with $f_{\|}$ are defined

Now lets calculate d_1, d_2, d_3, Ω and θ to calculate $m_{\|}$ and define L_{tan} that pass through $(x_{\|}, f_{\|}(x_{\|}))$

Hence

$$x_{f_{\|}+\delta} = d_3 cos(\angle d_1 d_3) cos(\theta) + x_{\|}$$

$$y_{f_{\|}+\delta} = d_3 cos(\angle d_1 d_3) sin(\theta) + f_{\|}(x_{\|})$$

$$x_{f_{\|}-\delta} = x_{\|} - d_2 cos(\angle d_1 d_2) cos(\theta)$$

$$y_{f_{\|}-\delta} = f_{\|}(x_{\|}) - d_2 cos(\angle d_1 d_2) sin(\theta)$$

and

$$d[(x_{\|-\delta}, f_{\|}(x_{\|-\delta})); (x_{\|+\delta}, f_{\|+\delta}(x_{\|+\delta}))] = q(t)_{E_{f_{\|}}/N_{f_{\|}}} \in (E_{f_{\|}}/N_{E_{f_{\|}}})^*$$

Considering that $L := [(x_1, x_2); (x_a, x_b)] \in B \times SB = F$

$$(x_{\|+\delta}, f_{\|}(x_{\|+\delta})) \in B$$

and

$$(x_{\|-\delta}, f_{\|}(x_{\|-\delta})) \in SB$$

With

$$B \subseteq E_{f_{\|}}$$

$$SB \subseteq N_{E_{f_{\|}}}$$

and by the inverse of the projection

$$\sigma^{-1}(x_{\|+\delta}, f_{\|}(x_{\|+\delta})) = (x_{\|CSB}, y_{\|CSB}) \in CSB_{(h_1,h_2)}$$

Hence the parallel transport of the connection ∇ is invertible

* $GL(E_x)$

Def. An automorphism is simply a bijective homomorphism of an object into itself.

Lets consider $E = \mathbb{C}$ and lets define its group of automorphisms (bijective linear transformations $\mathbb{C} \to \mathbb{C}$)

With

$J = e_{\Re} := 1$

* Group of J

Like Identity element is the only element that fulfill condition

$$e_{\Re}((x, f(x)) = (x, f(x))$$

hence the group of automorphism is the trivial group

*

Early was proved that $e_{\Re}$ is a bijection and is linear

With the set $GL(E) = \{e_{\Re}\}$ the group is $(GL(E); \circ)$ with the composition of functions

Like product by an scalar is defined with $e_{\Re}$ on E

Hence

$$e_{\Re}(x, f(x)) = (x, f(x)) \in CSB \subseteq E$$

with $\sigma[e_{\Re}(x, f(x))] = (x', f(x'))$

defines $e_{\Re}(x, f(x)) + e_{\Re}(x_a, f(x_a)) \in E/N_E$ (i.e. the quotient group)

Hence the map is linear for

$$\nabla(CSB_1 \cup CSB_2) = \nabla(CSB_1) \cup \nabla(CSB_2)$$

Hence $\exists$ a complex structure J on T_pM of E at each point

Like the quotient group defines a straight line (for E and T_pM) and

$detA = 0$ for parallel lines

and $detA \neq 0$ for other cases

Hence the Riemannian metric is preserved by parallel transport and J (with coefficients of the lines used like entries of the matrix)

Hence metric only can be preserved when $GL(E)$ is $(GL(E), \circ)$ (with $\circ$ the composition of functions)

* Holonomy group

With parallel transport

$$P_x : T_pM \to E$$

Hence

$$Hol_x(\nabla) = \{x_{GL} \in GL(E_x) | e_{\Re}(x_i, f(x_i)) \in CSB \quad \text{and} \quad x_i = \gamma(f(x_i)) \neq \gamma(f(x_j)) = x_j\} = \{e_{\Re}\}$$

*Holonomy group is a subset of Unitary group (U(n))

Like inner product is

$$v_1 \bullet_{\Re} v_2 = ||v_1|| \cdot ||v_2|| cos(\theta)$$

(scheme pag. 22)

$$v_1 = (ac - bd, ad + bc) = (a, b) \bullet_C (c, d) \quad ;$$

$$v_2 = (a'c' - b'd', a'd' + b'c') = (a', b') \bullet_{\mathbb{C}} (c', d')$$

$$\theta = \angle v_1 O v_2$$

where $|| \cdot ||$:= vector norm

Like $x = detA \in \Re$ is equivalent to a matrix 1×1 (i.e. $x \equiv [x]$)

Hence if

$$||v_i|| = detA$$

with $v_i \in \mathbb{C}$

Like vector norm

$$||v|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

is equal to the distance of a vector v to the origin; i.e.

$$||v|| = d(v,0)$$

Like

$$d(v,0) = q(t)$$

Hence $||v||$ is a linear transformation with $detA$ established by a system of linear equations of two points (a,b) defined by the complex product $\bullet_{\mathbb{C}}$

Now like

$$J := e_{\Re} \quad ;$$

$$J : V \to V$$

such that

$$e_{\Re}^2 = [Add_{map}(1,0)]^2 > -Add_{map}(1,0)$$

hence

$$(1+0)^2 > -(1+0)$$

i.e.

$$1 =_A -1$$

and $(1,0) = e_{\mathbb{C}} \in \mathbb{C}$

Like

$$e_{\Re}^2 = e_{\Re} \cdot e_{\Re} = [\bar{e_{\Re}}] \cdot [e_{\Re}] = A^{\dagger} A$$

with

$A :=$ matrix

$A^\dagger :=$ transpose conjugate

Hence the set of all matrix 1×1 with complex entries v_i that fulfill condition $A^\dagger A = I$

$$\mathrm{U}(1) = \{q(t) \in Q(t) | q(t) = detA = ||v|| = B = [q(t)]_{1\times 1} \quad ; \quad \text{s.t. } B^\dagger B = I\}$$

** $U(1)$ is a Lie group

Like the set that defines $U(1) = \{e_{\Re}\}$ with $e_{\Re}$ the identity of the Lie group $Q(t)$; hence $U(1)$ is a trivial group of Lie

** Algebraic structure of $U(1)$ consist of $n \times n$ skew-Hermitian matrices and Lie Algebra

Def. A Lie algebra is a vector space g over a field F together with a binary operation $[\cdot;\cdot] : g \times g \to g$ called the Lie bracket, satisfying the following axioms:

Bilinearity

$$[ax + by, z] = a[x, z] + b[y, z]$$

$$[z, ax + by] = a[z, x] + b[z, y]$$

for all scalars a, b in F and all elements x, y in g

The alternating property

$$[x, x] = 0$$

for all $x \in g$

The Jacobi identity

$$[x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0$$

for all x, y, z in g

Proof.

with

$$q(t)=detA=<(x_a,f(x_a));(x_b,f(x_b))>=x_af(x_b)-f(x_1)x_2\in Q(t)$$

Was proved early that $<();()>$ is bilinear and "alternating" ($<(x,f(x));(x,f(x)))>=$ 0)

$-0-$

Def. A skew-Hermitian matrix is an square matrix A that fulfill condition $A^{\dagger}=-A$. With the elements of the diagonal "imaginary (condition $-a=_A a$ or 0). If the matrix is real then $A^T=-A$).

$-0-$

If A is skew-Hermitian and Lie bracket of two $q(t)\in U(1)$ satisfies Jacobi identity

Hence $U(1)$ is a Lie algebra

** Homotopy classes of loops

Def. Before A homotopy between two continuous functions F and G from a topological space X to a topological space Y is defined to be a continuous function $H:X\times[0,1]\to Y$ from the product of the space X with the unit interval $[0,1]$ to Y such that $H(x,0)=f(x)$ and $H(x,1)=g(x)$ for all $x\in X$

Def. Now A homotopy between two continuous functions F and G from a topological space X to a topological space Y is defined to be a function $H:X\times[0,1]\to Y$ from the product of the space X with the unit interval $[0,1]$ to Y such that $H(x,0)=f(x)$ and $H(x,1)=g(x)$ for all $x\in X$

Proof.

Lets take

$x = \gamma(y)$

with

$x \in [0, 1]$

and lets consider the set

$Im[\gamma(y) = x = 0]$

and

$Im[\gamma(y) = x = 1]$

Besides consider the cartesian product with the set $Y := Dom[\gamma(y)]$

$Dom[\gamma(y) = 0] \times Im[\gamma(y) = 0]$

and

$Dom[\gamma(y) = 1] \times Im[\gamma(y) = 1]$

And lets apply next definition

Def. Given a set of n points:

$$(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$$

there exists a unique polynomial $P(x)$ of degree at most $n - 1$ such that:

$$P(x_i) = y_i$$

for each i

Def. The interpolating polynomial can be constructed as:

$$P(x) = \sum_{i=1}^{n} y_i \cdot L_i(x)$$

where each $L_i(x)$ is the Lagrange basis polynomial:

$$L_i(x) = \Pi_{j=1;j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

$$-0-$$

Hence

$(y_i, x_i) \in Dom[\gamma(y_i) = 0] \times Im[\gamma(y_i) = 0]$

with

$$y_i = f(x_i')$$

and

$$x_i' = f^{-1}(y_i)$$

establish

$$P(x_i') = y_i := F(x_i')$$

and

$(y_j, x_j) \in Dom[\gamma(y_j) = 1] \times Im[\gamma(y_j) = 1]$

with

$$y_j = f(x_j')$$

and

$$x_j' = f^{-1}(y_j)$$

establish

$$P(x_i') = y_i := G(x_i')$$

By

$$-0-$$

Def. Every polynomial function

$$P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$$

is continuous in the entire real line $\Re$

$$-0-$$

By the theorem above, this interpolating polynomial is automatically continuous across all real numbers,

$F(x')$ and $G(x')$

are continuous

Hence $F(x')$ and $G(x')$ are homotopic

Code in Python for the function "Homotopy" (Code in Copilot Microsoft Bing)

$$-0-$$

```
import numpy as np
import sympy as sp
import matplotlib.pyplot as plt

# 1. Definir gamma(y) por partes
y = sp.Symbol('y')
gamma1 = -sp.Rational(1,2)*sp.cos(y) + sp.Rational(1,2)
# tramo [0,0.5]
```

```
gamma2 =  sp.Rational(1,2)*sp.cos(y) + sp.Rational(1,2)
# tramo (0.5,1]

# 2. Definir f(x') = x'^2 + 2x' - 1
x_prime = sp.Symbol('x_prime')
f = x_prime**2 + 2*x_prime - 1

# 3. Evaluar gamma(y) en los intervalos definidos
y_vals1 = np.linspace(0, 0.5, 250)
y_vals2 = np.linspace(0.5, 1, 250)
gamma_func1 = sp.lambdify(y, gamma1, 'numpy')
gamma_func2 = sp.lambdify(y, gamma2, 'numpy')
x_gamma_vals1 = gamma_func1(y_vals1)
x_gamma_vals2 = gamma_func2(y_vals2)

# Unir resultados para b squeda de grupos
y_vals = np.concatenate([y_vals1, y_vals2])
x_gamma_vals = np.concatenate([x_gamma_vals1, x_gamma_vals2])

# Encontrar los valores de y donde gamma(y)     0 y gamma(y)     1
tol = 1e-3
y_group0 = [val for val, xval in zip(y_vals, x_gamma_vals) if abs(xval - 0) < to
y_group1 = [val for val, xval in zip(y_vals, x_gamma_vals) if abs(xval - 1) < to

# 4. Resolver f(x') = y para cada valor de y en los grupos (solo soluciones real
points_group0 = []
for yi in y_group0:
    sols = sp.solve(sp.Eq(f, yi), x_prime)
    for sol in sols:
        if sol.is_real:
            points_group0.append((float(sol.evalf()), float(f.subs(x_prime, sol)

points_group1 = []
for yi in y_group1:
    sols = sp.solve(sp.Eq(f, yi), x_prime)
    for sol in sols:
        if sol.is_real:
            points_group1.append((float(sol.evalf()), float(f.subs(x_prime, sol)

# 5. Construir polinomios interpoladores
poly_group0 = sp.interpolate(points_group0, x_prime) if len(points_group0) > 1 e
poly_group1 = sp.interpolate(points_group1, x_prime) if len(points_group1) > 1 e

print("Interpolating polynomial for x=0 group:", poly_group0)
print("Interpolating polynomial for x=1 group:", poly_group1)
```

```
# —— Graficar ——
plt.figure(figsize=(12, 8))

# Subplot 1: gamma(y) por partes
plt.subplot(3,1,1)
plt.plot(y_vals1, x_gamma_vals1, label='gamma(y) = -1/2 cos(y)+1/2', color='red'
plt.plot(y_vals2, x_gamma_vals2, label='gamma(y) = 1/2 cos(y)+1/2', color='blue'
plt.title("Gamma(y) piecewise definition")
plt.xlabel("y")
plt.ylabel("x = gamma(y)")
plt.legend()
plt.grid(True)

# Subplot 2: f(x')
x_vals = np.linspace(-5, 5, 400)
f_vals = [float(f.subs(x_prime, val)) for val in x_vals]
plt.subplot(3,1,2)
plt.plot(x_vals, f_vals, label="f(x') = x'^2 + 2x' - 1", color='green')
plt.title("Function f(x')")
plt.xlabel("x'")
plt.ylabel("f(x')")
plt.legend()
plt.grid(True)

# Subplot 3: Polinomios interpoladores
plt.subplot(3,1,3)
plt.plot(x_vals, f_vals, label="f(x')", color='green')
if poly_group0:
    poly0_func = sp.lambdify(x_prime, poly_group0, 'numpy')
    plt.plot(x_vals, poly0_func(x_vals), '--', label="Interpolating polynomial f
if poly_group1:
    poly1_func = sp.lambdify(x_prime, poly_group1, 'numpy')
    plt.plot(x_vals, poly1_func(x_vals), '--', label="Interpolating polynomial f
plt.title("Interpolating Polynomials for groups x=0 and x=1")
plt.xlabel("x'")
plt.ylabel("y")
plt.legend()
plt.grid(True)

plt.show()
```

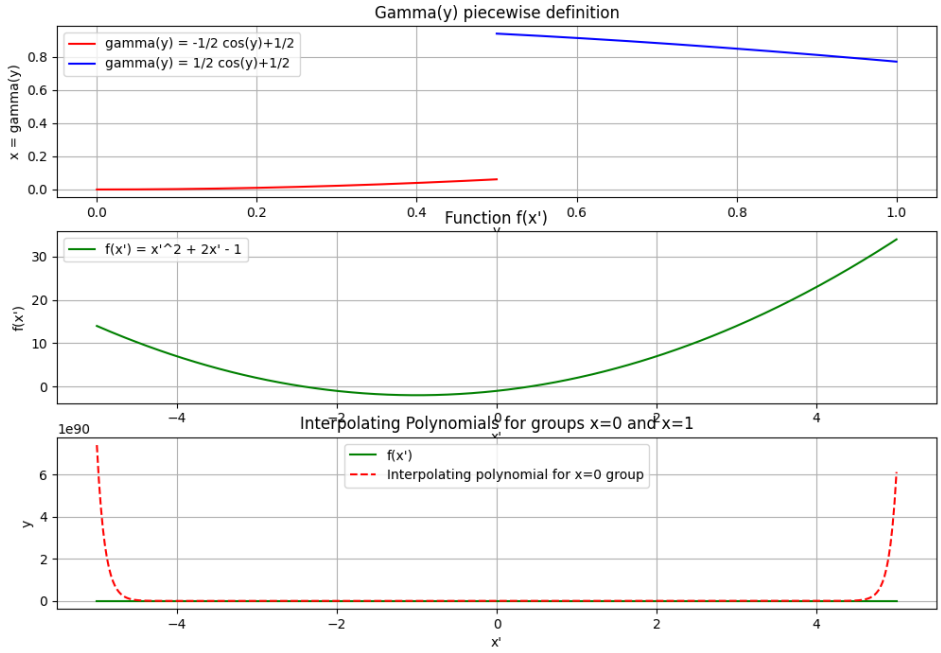
–0–

Results

Interpolating polynomial for x=0 group: 4.13783358175861e+42*x$_{p}rime$ * *63 + 2.25098146847668e + 44 * $x_{p}rime$ * *62 + 6.8463287759489e + 45 * $x_{p}rime$ * *61 + 1.32020411575301e + 47 * $x_{p}rime$ * *60 + 1.77549612972621e + 48 * $x_{p}rime$ * *59 + 1.54827829704743e + 49 * $x_{p}rime$ * *58 + 1.20573885079799e + 50 * $x_{p}rime$ * *57 + 8.03825900531997e + 50 * $x_{p}rime$ * *56 + 2.3267106066308e + 51 * $x_{p}rime$ * *55 + 1.85201487482572e + 52 * $x_{p}rime$ * *54 − 2.13262318919325e + 52 * $x_{p}rime$ * *53 + 7.25466028744863e + 53 * $x_{p}rime$ * *52 − 3.72198868177096e + 54 * $x_{p}rime$ * *51 + 1.25742456390187e + 55 * $x_{p}rime$ * *50 + 2.55765124935528e + 54 * $x_{p}rime$ * *49 + 1.93297972713123e + 56 * $x_{p}rime$ * *48 − 9.82318697970774e + 56 * $x_{p}rime$ * *47 + 2.34086299342982e + 57 * $x_{p}rime$ * *46 + 3.54357267747422e + 57 * $x_{p}rime$ * *45 + 3.68112762146602e + 58 * $x_{p}rime$ * *44 + 2.97830930270155e + 56 * $x_{p}rime$ * *43 + 6.60275703782068e + 57 * $x_{p}rime$ * *42 − 4.18044621653755e + 59 * $x_{p}rime$ * *41 + 1.06349328740649e + 60 * $x_{p}rime$ * *40 − 1.02994418236877e + 60 * $x_{p}rime$ * *39 − 2.01784837237467e + 59 * $x_{p}rime$ * *38 − 3.58095206355439e + 60 * $x_{p}rime$ * *37 + 5.0563341774794e + 60 * $x_{p}rime$ * *36 − 7.11269492904381e + 58 * $x_{p}rime$ * *35 + 6.8815306197924e + 60 * $x_{p}rime$ * *34 − 4.64035684441979e + 60 * $x_{p}rime$ * *33 − 7.47795106562544e + 60 * $x_{p}rime$ * *32 − 1.09199906455639e + 61 * $x_{p}rime$ * *31 − 2.5341304120131e + 60 * $x_{p}rime$ * *30 + 3.87586454762891e + 60 * $x_{p}rime$ * *29 − 2.48492811017806e + 60 * $x_{p}rime$ * *28 − 2.87737675185584e + 59 * $x_{p}rime$ * *27 + 1.72511107415847e + 59 * $x_{p}rime$ * *26 + 3.55403732327502e + 60 * $x_{p}rime$ * *25 + 2.8237228378175e + 60 * $x_{p}rime$ * *24 + 1.62724792031917e + 60 * $x_{p}rime$ * *23 + 2.77442705375447e + 59 * $x_{p}rime$ * *22 + 2.62749712815452e + 59 * $x_{p}rime$ * *21 − 3.21203642340214e + 58 * $x_{p}rime$ * *20 + 6.79645250156472e + 58 * $x_{p}rime$ * *19 + 7.40779610163179e + 57 * $x_{p}rime$ * *18 − 3.18944648132007e + 56 * $x_{p}rime$ * *17 + 1.66482293903323e + 57 * $x_{p}rime$ * *16 + 1.94483262233012e + 56 * $x_{p}rime$ * *15 − 8.69361972352534e + 55 * $x_{p}rime$ * *14 − 2.09992296695896e + 55 * $x_{p}rime$ * *13 − 1.47562558915407e + 54 * $x_{p}rime$ * *12 − 1.1037260365123e + 54 * $x_{p}rime$ * *11 − 7.87574002324877e + 52 * $x_{p}rime$ * *10 − 3.31117810953689e + 52 * $x_{p}rime$ * *9 − 3.81159627015899e + 51 * $x_{p}rime$ * *8 − 1.5842677748667e + 51 * $x_{p}rime$ * *7 + 1.39573406365101e + 50 * $x_{p}rime$ * *6 − 2.79512188139535e + 49 * $x_{p}rime$ * *5 + 3.0657280439324e + 48 * $x_{p}rime$ * *4 − 1.86969447744481e + 47 * $x_{p}rime$ * *3 + 9.14330553139756e + 45 * $x_{p}rime$ * *2 − 3.24754601953602e + 44 * $x_{p}rime$ + 5.46629594221796e + 42$Interpolating polynomial for x = 1 group : None$

[Execution Finished]

Code "Homotopy" in matplotlib



Pseudocode of Homotopy funtion (in spanish) on Copilot Microsoft Bing

```
funci n homotopy(rango_y):
    # Paso 1: Definir una funci n gamma(y) por tramos
    #   - F rmula 1 en intervalo A
    #   - F rmula 2 en intervalo B

    # Paso 2: Definir una funci n problema f(x')

    # Paso 3: Evaluar gamma(y) en el rango dado
    para cada y en rango_y:
        calcular x = gamma(y)
        guardar par (y, x)

    # Paso 4: Identificar subconjuntos de valores de y
    grupo0 = { y | gamma(y)      0 }
    grupo1 = { y | gamma(y)      1 }

    # Paso 5: Para cada grupo, resolver la ecuaci n
```

```
    #   f(x') = y
    # para encontrar x'
    puntos_grupo0 = []
    para cada y en grupo0:
        soluciones = resolver f(x') = y
        conservar solo soluciones reales
        agregar (x', f(x')) a puntos_grupo0

    puntos_grupo1 = []
    para cada y en grupo1:
        soluciones = resolver f(x') = y
        conservar solo soluciones reales
        agregar (x', f(x')) a puntos_grupo1

    # Paso 6: Construir polinomios interpoladores
    si puntos_grupo0 tiene m s de un punto:
        polinomio0 = interpolar(puntos_grupo0)
    si puntos_grupo1 tiene m s de un punto:
        polinomio1 = interpolar(puntos_grupo1)

    # Paso 7: Devolver resultados
    retornar polinomio0, polinomio1
fin funci n
```

Pseudocode of Homotopy funtion (in english) on Copilot Microsoft Bing

```
function homotopy(y_range):
    # Step 1: Define gamma(y) as a piecewise function
    #   gamma1(y) applies in interval A
    #   gamma2(y) applies in interval B

    # Step 2: Define problem function f(x')

    # Step 3: Evaluate gamma(y) over the given range
    for each y in y_range:
        compute x = gamma(y)
        store (y, x)

    # Step 4: Identify subsets of y
    group0 = { y | gamma(y)     0 }
    group1 = { y | gamma(y)     1 }

    # Step 5: For each group, solve equation
    #   f(x') = y
    # to find x'
    points_group0 = []
```

```
    for each y in group0:
        solutions = solve f(x') = y
        keep only real solutions
        add (x', f(x')) to points_group0

    points_group1 = []
    for each y in group1:
        solutions = solve f(x') = y
        keep only real solutions
        add (x', f(x')) to points_group1

    # Step 6: Build interpolating polynomials
    if points_group0 has more than one point:
        poly0 = interpolate(points_group0)
    if points_group1 has more than one point:
        poly1 = interpolate(points_group1)

    # Step 7: Output results
    return poly0, poly1
end function
```

Hence the class of homotopy is the set of functions defined by the "interpolator polynomium"

$$[f_i] := Hom(f_i) = \{f_i \quad | \quad f_i \text{ is homotopic to } f_j \quad \text{s.t.} \quad x_i = \gamma_1(y_i) \quad \text{and} \quad x_j = \gamma_2(y_j)\}$$

Like $Hol_x(\nabla) = \{e_{\Re}\}$ define a set with

$$x_i = \gamma(f(x_i')) \neq \gamma(f(x_j')) = x_j$$

and $U(1) := \{q(t) \in Q(t)\}$ a set where product between q(t) is defined where $q(t) = e_{\Re}$

with

$$\gamma(y) : Y \to [0, 1]$$

Hence the fundamental group is

$$(\pi_x(Y), \circ_f)$$

where

$\circ_f :=$ is the composition of functions

and the subindex x define the point of $x \in [0,1]$ where $y = f(x')$

Hence $U(1)$ is an algebraic structure

and

$Hol_x(\nabla)$ a set

Like the set of element that define $U(1) = e_{\Re}$

Hence

$$U(1) := (Hol_x(\nabla), \cdot_A)$$

where

$\cdot_A :=$ the product between matrix

Hence the Riemannian manifold $\mathbb{C}$ is a Kähler manifold

** Kähler class and de Rham Cohomology

Def. De Rham Cohomology group in grade k is defined like the quotient:

$$H^k_{dR}(M) = \frac{Z^k(M)}{B^k(M)}$$

where

$$H^k_{dR}(M) \in \{H^k_{dR}(M)\}_{k \geq 0}$$

establish an element in "de Rham" cohomology of a manifold

and

$Z^k(M) :=$ Closed $k-$ forms of grade k where (before)

$$Z^k(M) = \{\omega \in \Omega^k(M) \quad | \quad d\omega = 0\}$$

$B^k(M) :=$ Exact $k-$ forms of grade k where (before)

$$B^k(M) = \{\omega \in \Omega^k(M) \quad | \quad \exists \eta \in \Omega^k(M) \quad ; \quad \omega = d\eta\}$$

with

$\Omega^k(M) :=$ the space of all $k-$forms

Proof.

Lets remember that ω the form of degree $k = 2$ is

*) the smooth section

$$\sigma : CSB_{(h_1,h_2)} \to E$$

of the

*) *2 - exterior power* of the cotangent bundle

$$a \wedge b := \begin{cases} (a,b) \in X \times X \subseteq CSB_{(h_1,h_2)} & ; \\ ae_i + be_j = c \in R = T^2(X) & ; \\ c_1 - c_2 \in I \quad \text{such that} \quad c = xr(\text{rx}) & \end{cases}$$

And the exterior derivative of the 2-form ω is

*) *2 - exterior power* of the cotangent bundle

$$
d(a\wedge b) := \left\{ \begin{array}{rl}
(a,b) \in X \times X \subseteq CSB_{(h_1,h_2)} & ; \\
b = f(a) & ; \\
m_{\parallel} = \frac{y_{\parallel_{\delta+}} - y_{\parallel_{\delta-}}}{x_{\parallel_{\delta+}} - x_{\parallel_{\delta-}}} & ; \\
y = m_{\parallel}(x-a) + b := \text{tangent line} \in \Re^2 & ; \\
z = \frac{y-f(a)}{x-a} \in \Re^3 & ; \\
z = m_{\parallel} \text{ line defined by interseccion of planes} & ; \\
\text{intersecion line is part of: } y = (x-a) + b := \text{tangent plane} \in \Re^3 & ; \\
ae_i + be_j = c \in R = T^2(X) & ; \\
c_1 - c_2 \in I \quad \text{such that} \quad c = xr(\text{or rx}) &
\end{array} \right.
$$

with (a,b) on the smooth section

$$\sigma : CSB_{(h_1,h_2)} \to E$$

with

$$f(a) \in \Re^n$$

a function that is only dependent of "a"

when

$$z = 0 \quad ;$$

$$d(x_1^* \wedge x_2^*) = 0$$

i.e. the differential 2-form is closed

with $d\omega := d(a \wedge b)$

the set of the closed 2-form with $M = \mathbb{C}$ is

$$Z^2(\mathbb{C}) = \{(a,b) \in \Re \times \Re \subseteq CSB_{(h_1,h_2)} \quad | \quad d\omega := d(a \wedge b) \text{ and } z = 0\}$$

Like the derivative $df(a) := m_{\parallel}$ hence $d(a\wedge b)$ establish the pair $(a, m_{\parallel;(a,f(a))})$

Hence

$$B^2(\mathbb{C}) = \{(a,b) \in \Re \times \Re \subseteq CSB_{(h_1,h_2)} \quad | \quad b = m_{\|;(a,f(a))}\}$$

with $\eta := a \wedge b$

$\exists \quad \eta$ a ω 2-form

* Quotient group

* $B(\mathbb{C})$ is normal in $Z(\mathbb{C})$

Proof.

Lets take $(a, m_{\|;(a,f(a))}) \in B(\mathbb{C})$ and consider $(a, f(a)) \in Z(\mathbb{C})$ such that

$$(a, f(a)) + (a, m_{\|;(a,f(a))}) \in (a, f(a)) + B(\mathbb{C})$$

and

$$(a, m_{\|;(a,f(a))}) + (a, f(a)) \in B(\mathbb{C}) + (a, f(a))$$

Like $(a,b) + (c,d) = L[(a,b),(c,d)]$ and by $\equiv_{AND}$

$$(a, f(a)) + (a, m_{\|;(a,f(a))}) = (a, m_{\|;(a,f(a))}) + (a, f(a))$$

Hence

$$(a, f(a)) + B(\mathbb{C}) = B(\mathbb{C}) + (a, f(a))$$

else if

$$|B(\mathbb{C})| < |Z(\mathbb{C})|$$

hence

$$B(\mathbb{C}) \subseteq Z(\mathbb{C})$$

hence like $\mathbb{C}$ is a group under addition to be a vector space

$B(\mathbb{C})$ is normal in $Z(\mathbb{C})$

* Quotient Group

Hence the product of two lateral classes is

$((a, f(a)) + B(\mathbb{C}))((a, f(a)) + B(\mathbb{C})) = L_{(a,f(a));B(\mathbb{C})} \cdot L_{(a,f(a));B(\mathbb{C})}$

$= |L_{(a,f(a));B(\mathbb{C})}| \cdot |L_{(a,f(a));B(\mathbb{C})}| cos(\theta)$

where

$\theta :=$ angle between lines

Hence

$$H_{dR} = Z(\mathbb{C})/B(\mathbb{C}) = \{(a, f(a)) + B(\mathbb{C}) \quad | \quad (a, f(a)) \in Z(\mathbb{C})\}$$

is the quotient group

Like the grade only can be 2, hence is not used

Def. Kähler class is an homology class of de Rham represented by the Kähler form ω

Formally

$$[\omega] \in H^2(X, R)$$

where X is a complex manifold

Like $\omega :=$ 2-form hence

$$[\eta] \in Z^2(\mathbb{C})$$

7 * Complex Variety and Kähler manifold

Def. Every smooth complex projective variety is a Kähler manifold.

Def. Let k be an algebraically closed field and let P^n be the projective n-space over k. Let f in $k[x_0, ..., x_n]$ be a homogeneous polynomial of degree d. It is not well-defined to evaluate f on points in P^n in homogeneous coordinates.

However, because f is homogeneous, meaning that $f(\lambda x_0, ..., \lambda x_n) = \lambda^d f(x_0, ..., x_n)$, it does make sense to ask whether f vanishes at a point $[x_0 : ... : x_n]$. For each set S of homogeneous polynomials (that generate a prime ideal), define the zero-locus of S to be the set of points in P^n on which the functions in S vanish:

$$Z(S) = \{x \in \mathbf{P}^n \mid f(x) = 0 \text{ for all } f \in S\}$$

A subset V of P^n is called a projective algebraic set if $V = Z(S)$ for some S. An irreducible projective algebraic set is called a projective variety

Def. Before Let k be an algebraically closed field, and V be a finite-dimensional vector space over k. The symmetric algebra of the dual vector space V^* is called the polynomial ring on V and denoted by $k[V]$. It is a naturally graded algebra by the degree of polynomials. Let V be a finite-dimensional vector space over a field k. The scheme over k defined by $Proj(k[V])$ is called projectivization of V.

The projective n-space on k is the projectivization of the vector space $\mathbb{A}_k^{n+1}$.

Def. Now Let k be an algebraically closed field, and V be a finite-dimensional vector space over k. The dual vector space V^* define the polynomial ring on V by presheaves. It span[S] is a naturally graded algebra by the degree of polynomials.

Let V_{P_k} be a finite-dimensional vector space over a field k. The scheme over V^* defined by $Proj(k[V])$ is called projectivization of V.

The projective n-space on k is the projectivization of the vector space $\mathbb{A}_k^{n+1}$.

Def. A scheme is a locally ringed space X admitting a covering by open sets U_i, such that each U_i (as a locally ringed space) is an affine scheme.

Def. A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that all stalks of $\mathcal{O}_X$ are local rings (i.e. they have unique maximal ideals). Note that it is not

required that $\mathcal{O}_X(U)$ be a local ring for every open set U; in fact, this is almost never the case.

Def. The stalk of a sheaf $\mathcal{F}$ at x, usually denoted $\mathcal{F}_x$, is:

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$$

Here the direct limit is indexed over all the open sets containing x, with order relation induced by reverse inclusion ($U \leq V$, if $U \supseteq V$). By definition (or universal property) of the direct limit, an element of the stalk is an equivalence class of elements $f_U \in \mathcal{F}(U)$, where two such sections f_U and f_V are considered equivalent if the restrictions of the two sections coincide on some neighborhood of x.

Def. A ringed space $(X, \mathcal{O}_X)$ is a topological space X together with a sheaf of rings $\mathcal{O}_X$ on X. The sheaf $\mathcal{O}_X$ is called the structure sheaf of X.

Def. An affine scheme is a locally ringed space isomorphic to the spectrum $\mathrm{Spec}(R)$ of a commutative ring R.

Def. In commutative algebra, the prime spectrum (or simply the spectrum) of a commutative ring R is the set of all prime ideals of R, and is usually denoted by $\operatorname{Spec} R$; in algebraic geometry it is simultaneously a topological space equipped with a sheaf of rings.

Def. Before Axiomatically, a sheaf is a presheaf that satisfies both of the following axioms:

1. (Locality) Suppose U is an open set, $\{U_i\}_{i \in I}$ is an open cover of U with $U_i \subseteq U$ for all $i \in I$, and $s, t \in \mathcal{F}(U)$ are sections. If $s_i|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

2. (Gluing) Suppose U is an open set, $\{U_i\}_{i \in I}$ is an open cover of U with $U_i \subseteq U$ for all $i \in I$, and $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$ is a family of sections. If all pairs of sections agree on the overlap of their domains, that is, if $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Def. Now Axiomatically, a sheaf is a presheaf that satisfies both of the following axioms:

1. (Locality) Suppose U is a closed set, $\{U_i\}_{i \in I}$ is an open cover of U with $U_i \subseteq U$ for all $i \in I$, and $s, t \in \mathcal{F}(U)$ are sections. If $s_i|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

2. (Gluing) Suppose U is an closed set, $\{U_i\}_{i\in I}$ is an open cover of U with $U_i \subseteq U$ for all $i \in I$, and $\{s_i \in \mathcal{F}(U_i)\}_{i\in I}$ is a family of sections. If all pairs of sections agree on the overlap of their domains, that is, if $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Def. Before Presheaves (category theory)

Let X be a topological space. A presheaf $\mathcal{F}$ of sets on X consists of the following data:

* For each open set $U \subseteq X$, there exists a set $\mathcal{F}(U)$. This set is also denoted $\Gamma(U, \mathcal{F})$. The elements in this set are called the sections of $\mathcal{F}$ over U. The sections of $\mathcal{F}$ over X are called the global sections of $\mathcal{F}$.

* For each inclusion of open sets $V \subseteq U$, a function $\mathrm{res}^U_V \colon \mathcal{F}(U) \to \mathcal{F}(V)$. In view of many of the examples below, the morphisms res^U_V are called restriction morphisms. If $s \in \mathcal{F}(U)$, then its restriction $\mathrm{res}^U_V(s)$ is often denoted $s|_V$ by analogy with restriction of functions.

The restriction morphisms are required to satisfy two additional (functorial) properties:

* For every open set U of X, the restriction morphism $\mathrm{res}^U_U \colon \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity morphism on $\mathcal{F}(U)$.

* If we have three open sets $W \subseteq V \subseteq U$, then the composite $\mathrm{res}^V_W \circ \mathrm{res}^U_V = \mathrm{res}^U_W$.

Def. Now Presheaves (category theory)

Let X be a topological space. A presheaf $\mathcal{F}$ of sets on X consists of the following data:

* For each $U \subseteq X$ (open or closed), there exists a set $\mathcal{F}(U)$. This set is also denoted $\Gamma(U, \mathcal{F})$. The elements in this set are called the sections of $\mathcal{F}$ over U. The sections of $\mathcal{F}$ over X are called the global sections of $\mathcal{F}$.

* For each inclusion of $V \subseteq U$, a function $\mathrm{res}^U_V \colon \mathcal{F}(U) \to \mathcal{F}(V)$. In view of many of the examples below, the morphisms res^U_V are called restriction morphisms. If $s \in \mathcal{F}(U)$, then its restriction $\mathrm{res}^U_V(s)$ is often denoted $s|_V$ by analogy with restriction of functions.

The restriction morphisms are required to satisfy two additional (functorial) properties:

* For every $U \subseteq X$, the restriction morphism $\mathrm{res}_U^U \colon \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity morphism on $\mathcal{F}(U)$.

* If we have three sets $W \subseteq V \subseteq U$, then the composite $\mathrm{res}_W^V \circ \mathrm{res}_V^U = \mathrm{res}_W^U$.

Proof.

Lets take

$F(x)$ and $G(x)$

and consider

$-0-$

Prop. A general interpolating polynomial is not homogeneous, since it typically contains terms of different degrees. However, every polynomial can be decomposed into a sum of homogeneous polynomials. More precisely, if

$$P(x_1, x_2, \cdots, x_n) \in K[x_1, x_2, \cdots, x_n],$$

then there exists a unique decomposition

$$P = P_0 + P_1 + \cdots + P_m,$$

where each P_k is homogeneous of degree k.

Thus, although the interpolating polynomial itself is not homogeneous, it can always be expressed as a sum of homogeneous components. This decomposition is canonical and reflects the fact that the space of polynomials is graded by degree.

$-0-$

Lets define the set of homogeneous polynomials

$$S = \{P_k \in K[x_1, ..., x_n] \quad | \quad k := \text{ degree and } P_k := \text{ is homogenoeus } \}$$

Now lets establish the " prime ideal" of **S**

$-0-$

Def. Given a ring R, a left ideal is a subset I of R that is a subgroup of the additive group of R that is closed under left multiplication by elements of R; that is, $0 \in I$ and for every $r \in R$ and every , one has

* $x + y \in I$
* $-x \in I$
* $rx \in I$

In other words, a left ideal is a left submodule of R considered as a left module over itself.

A right ideal is defined similarly, with the condition $rx \in I$ replaced by $xr \in I$. A two-sided ideal is a left ideal that is also a right ideal.

If the ring is commutative, the definitions of left, right, and two-sided ideal coincide, and one talks simply of an ideal.

Def. An ideal I of a commutative ring R is prime if it has the following two properties:

* If a and b are two elements of R such that their product ab is an element of P, then a is in I or b is in I

* I is not the whole ring R

$-o-$

Proof.

With

$$S = \{P_k \in K[x_1, ..., x_n] \quad | \quad k := \text{ degree and } P_k := \text{ is homogenoeus } \}$$

the set of homogeneous polynomials

- Closure of S

If $P_{k1}, P_{k2} \in S$;

$P_{k1} + P_{k2} \in S$

$P_{k1} \cdot P_{k2} \in S$

- S is an abelian group under addition

$$(P_{k1} + P_{k2}) + P_{k3} = P_{k1} + P_{k2} + P_{k3} = P_{k1} + (P_{k2} + P_{k3}); \quad (\text{ associative})$$

$$P_{k1} + P_{k2} = P_{k2} + P_{k1}; \quad (\text{commutative})$$

With $P_k = 0$

$$P_k + 0 = P_k; \quad (\text{additive identity})$$

If $P_k + (-P_k) = 0$ hence $\exists - P_k$ for each $P_k \in S$ (additive inverse)

* S is a monoid under multiplication, meaning that:

$$(P_{k1} \cdot P_{k2}) \cdot P_{k3} = P_{k1} \cdot P_{k2} \cdot P_{k3} = P_{k1} \cdot (P_{k2} \cdot P_{k3}); \quad (\text{associative})$$

With $P_k = 1$

$P_k \cdot 1 = P_k$ and $1 \cdot P_k = P_k$ (multiplicative identity)

* Multiplication is distributive with respect to addition, meaning that

$$P_{k1} \cdot (P_{k2} + P_{k3}) = (P_{k1} \cdot P_{k2}) + (P_{k1} \cdot P_{k3}); \quad (\text{left distributivity})$$

$$(P_{k2} + P_{k3}) \cdot P_{k1} = (P_{k2} \cdot P_{k1}) + (P_{k3} \cdot P_{k1}); \quad (\text{right distributivity})$$

Hence S is a ring

* S is an ideal

-Left ideal

By 1 S is an abelian group under addition

Lets consider $I \subseteq S$

If $0 \in I$

and

$P_{x1} + P_{x2} \in I$

$-P_x \in I$

$P_{x1} \cdot P_{x2} \in I$

Hence $I \subseteq S$ is a left ideal

- Right ideal

If $P_{x2} \cdot P_{x1} \in I$

Hence $I \subseteq S$ is a right ideal

Hence I is an ideal on S

* I is prime

With S a commutative ring

If $P_{xr} \in S$ but $P_{xr} \notin I$ hence $I \subseteq S$

If $P_{xa} \notin I$ or $P_{xb} \notin I$ with $P_{xa}, P_{xb} \in S$

Hence $P_{xa} \cdot P_{xb} \notin I$

* Prime spectrum

Lets consider S a commutative ring and $P(S)$ its power set

Now lets consider $I \subset S$ (an ideal ring prime) such that $I \subseteq P(S)$

Lets called last set like $I_{P(S)}$

If $I_{P(S)} \subseteq S$

Hence

$$Spec(S) = \{ \quad I_{P(S)} \subseteq S \quad | \quad I_{P(S)} \text{ is a prime ideal} \quad \}$$

Is the spectrum of the ring S

* Affine scheme and locally ringed space

Proof. First lets take $Q(t)$ the dual of $\mathbb{C}$ and lets define the category of $Q(t)$ with $res_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$;

First lets take into account next definitions

$-0-$

Def. There are many equivalent definitions of a category. One commonly used definition is as follows. A category $\mathcal{C}$ consists of

*a class $\text{ob}(\mathcal{C})$ of objects

*a class $\text{mor}(\mathcal{C})$ of morphisms or arrows,

*a domain or source class function $\text{dom} : \text{mor}(\mathcal{C}) \to \text{ob}(\mathcal{C})$,

*a codomain or target class function $\text{cod} : \text{mor}(\mathcal{C}) \to \text{ob}(\mathcal{C})$,

Def. In set theory and its applications throughout mathematics, a class is a collection of mathematical objects (often sets) that can be unambiguously defined by a property that all its members share.

$-0-$

See: Q1 (in spanish)

See: Q2 (in spanish

Element	Definition in OOP	Mathematical analogy
Class	Template grouping attributes and methods	Set defined by a common property
Object	Concrete instance of a class	Element satisfying the class property
Attribute	Internal variable of the object	Parameter or logical property
Method	Function operating on the object	Operation defined on set elements
Encapsulation	Hiding internal details	Restriction of access to internal component
Inheritance	Reuse of class structure	Inclusion of subsets with extended properti
Polymorphism	Ability to use methods in multiple forms	Functions defined for different input types

See: Q3 (in spanish)

See: Q4 (in spanish)

See: Q5 (in spanish)

See: Q6 (in spanish)

Def. Object-Oriented Programming is a design and coding approach that models software as a set of objects interacting with each other. Each object belongs to a class, which defines its properties (attributes) and behaviors (methods).

7.0.1 Key components:

See: Q7 (in spanish)

Def. In mathematical logic, a **logical structure** (also called a *model*) is a formal set that gives meaning to the symbols of a logical language. It consists of:

$$A = (U, \{R_i\}, \{f_j\}, \{c_k\})$$

Where:

- U is the **universe** or set of elements (domain).
- $\{R_i\}$ are the **relations** defined over U.
- $\{f_j\}$ are the **functions** defined over U.
- $\{c_k\}$ are the **constants** of the language.

This structure allows us to **interpret logical formulas** and determine whether they are true or false in that model.

Now lets take next questions to establish the formal proof about "Math Class is equal to a logic structure and equal to an OOP"

See: Q8 (in spanish)
See: Q9 (in spanish)
See: Q10 (in spanish)
See: Q11 (in spanish)
See: Q12 (in english)
See: Q13 (in english)
See: Q14 (in english)
See: Q15 (in english)
See: Q16 (in english)
See: Q17 (in english)
See: Q18 (in english)
See: Q19 (in english)
See: Q20 (in english)
See: Q21 (in english)
See: Q22 (in english)
See: Q23 (in english)
See: Q24 (in english)
See: Q25 (in english text below)
See: Q26 (in english)
See: Q27 (in spanish)
See: Q28 (in english)
See: Q29 (in english)

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8 Equivalence of Mathematical Classes, Logical Structures, and OOP Classes

author: José Rodrigo Alejandro Martínez Díaz (with Copilot Microsoft Bing)

Abstract

We present a maximally strict formalization of the claim that mathematical classes, logical structures, and object-oriented programming (OOP) classes are equivalent frameworks for describing structured collections of objects. The proof is separated into three layers: (1) syntactic Hilbert derivation, (2) semantic model-theoretic interpretation, and (3) programming analogy. A "Resumen" section expands with explicit mini-proofs of equality and substitution axioms, and a worked-out example of group theory in Hilbert style.

9 Introduction

Mathematical classes are defined by axioms, logical structures interpret those axioms in first-order logic, and OOP classes implement them in programming.

We aim to show their equivalence. To ensure maximal strictness, we separate syntactic derivations from semantic reasoning and programming analogies.

10 Definitions

Logical Structure. A structure for a first-order language L is

$$\mathcal{A} = (U, \{R_i^{\mathcal{A}}\}, \{f_j^{\mathcal{A}}\}, \{c_k^{\mathcal{A}}\})$$

where U is the universe (or domain of the logical structure; the set of objects); R_i^A are the relations in the structure (relation among objects); f_j^A are the functions in the structure (function on objects) and c_k^A are the constants (distinguished elements of the set).

Mathematical Class.

$$\mathcal{C} = \{X \mid X \models P\}$$

where P is a set of axioms in L.

OOP Class.

$$\text{Class} = (U, A, M, C)$$

with instances U, attributes A, methods M, constants C.

11 Layer One: Strict Hilbert Derivation

We work purely syntactically.

11.1 Axiom schemata

1. $\varphi \to (\psi \to \varphi)$ (axiom of implication introduction)
2. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$ (axiom of distributivity of implication)
3. $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$ (contraposition axiom)
4. Quantifier schemata:
   - $\forall x\, \varphi(x) \to \varphi(t)$
   - $\varphi(t) \to \exists x\, \varphi(x)$
5. Equality schemata:
   - $x = x$ (reflexivity)
   - $x = y \to y = x$ (symmetry)
   - $(x = y \wedge y = z) \to x = z$ (transitivity)
   - $x = y \to (\varphi(x) \to \varphi(y))$ (substitution)

11.2 Derivations

- From $\forall x\, R(x)$, derive $R(t)$ via quantifier axiom + Modus Ponens.
- From $\forall x \exists y\, f(x) = y$, derive $\exists y\, f(t) = y$.
- From equality axiom $c = c$, derive $\exists x\, (x = c)$.

Thus relations, functions, and constants are introduced syntactically.

12 Layer Two: Semantic Model Theory

Interpretations of R_i, f_j, c_k in a model $\mathcal{A}$ guarantee that $\mathcal{A} \models P$. The class $\mathcal{C}$ is the set of all such models. This layer is formal but semantic.

13 Layer Three: Programming Analogy

Mapping:

$$R_i^{\mathcal{A}} \mapsto \text{method}, \quad f_j^{\mathcal{A}} \mapsto \text{method}, \quad c_k^{\mathcal{A}} \mapsto \text{constant}$$

Attributes A represent data fields; methods M enforce axioms; constants C are distinguished values. This layer is informal.

14 Theorem and Proof

Theorem. For any set of axioms P in L, the class $\mathcal{C}$, logical structure $\mathcal{A}$, and OOP class Class correspond.

Proof.

1. By Hilbert derivation, relations, functions, and constants are introduced syntactically.
2. By model theory, these derivations correspond to structures satisfying P.
3. By analogy, OOP classes implement the same components.

Therefore:

$$\mathcal{C} \equiv \mathcal{A} \equiv \text{Class}$$

15 Resumen: Expanded Mini-Proofs

To make the document fully formal, we include explicit derivations:

15.1 Equality Reflexivity

1. Axiom: $x = x$. 2. By Generalization: $\forall x\, (x = x)$.

15.2 Equality Symmetry

1. Axiom: $x = y \to y = x$. 2. Assume $x = y$. 3. By Modus Ponens: infer $y = x$.

15.3 Equality Transitivity

1. Axiom: $(x = y \land y = z) \to x = z$. 2. Assume $x = y$ and $y = z$. 3. By conjunction introduction: $(x = y \land y = z)$. 4. By Modus Ponens: infer $x = z$.

15.4 Substitution

1. Axiom: $x = y \to (\varphi(x) \to \varphi(y))$. 2. Assume $x = y$ and $\varphi(x)$. 3. By Modus Ponens twice: infer $\varphi(y)$.

16 Worked Example: Group Axioms

Language L: binary function symbol $\cdot$, constant e, variables x, y, z.

16.1 Axioms

1. $\forall x \forall y \forall z\ ((x \cdot y) \cdot z = x \cdot (y \cdot z))$ (associativity).
2. $\forall x\ (e \cdot x = x \land x \cdot e = x)$ (identity).
3. $\forall x\ \exists y\ (x \cdot y = e \land y \cdot x = e)$ (inverse).

16.2 Derivation of Identity

1. From axiom (2): $\forall x\ (e \cdot x = x)$. 2. By quantifier axiom: $e \cdot a = a$. 3. From axiom (2): $\forall x\ (x \cdot e = x)$. 4. By quantifier axiom: $a \cdot e = a$. 5. By conjunction introduction: $(e \cdot a = a) \land (a \cdot e = a)$. 6. By generalization: $\forall a\ ((e \cdot a = a) \land (a \cdot e = a))$. Thus e is an identity element.

16.3 Derivation of Inverse

1. From axiom (3): $\forall x\ \exists y\ (x \cdot y = e \land y \cdot x = e)$. 2. By quantifier axiom: $\exists y\ (a \cdot y = e \land y \cdot a = e)$. Thus every element has an inverse.

16.4 Conclusion

The Hilbert derivations show that the group axioms syntactically guarantee identity and inverses. This concrete case illustrates the general framework.

Concept	Logical Structure (Mathematics/Logic)	Object-Oriented Program
Universe / Objects	U: the set of elements in the model	The set of instantiated ob
Relations	$\{R_i\}$: predicates connecting elements	Methods that compare or
Functions	$\{f_j\}$: operations on the universe	Methods that operate on
Constants	$\{c_k\}$: fixed elements of the universe	Fixed values defined in th
Language / Definition	Formal logical language interpreted in the structure	Class definition (attribute
Example	$N = (N, +, \cdot, 0, 1)$	Class Metric { attributes:

17 Conclusion

We have separated the proof into three layers. The Hilbert derivation is maximally strict syntactically. The semantic model theory layer is formal but not syntactic. The programming analogy is informal. The "Resumen" now includes explicit Hilbert derivations of equality axioms, and the group axioms example demonstrates the framework in practice.

$-0-$

Now lets take into account next table (see Q27 (in english))

* **Equivalence Table**

Now lets establish the class of all objects $q(t) \in Q(t)$ such that $Q(t)$ is a Lie group and the duals space of the vector space $\mathbb{C}$

Remember that

$$q(t) = (\frac{n}{q_{e^-}})(\frac{h\nu(dt)^2}{\frac{1}{2}m_{e^-}(dx)^2})$$

Hence

* Class of objects

$$ob(C) = \{U_{q(t)} \subseteq Q(t) \quad | \quad Q(t) := \text{ Lie group and a dual space } \}$$

With U an open set on $Q(t)$

* Class of morphism

If $V \subseteq U$ is an open set

Hence

$$mor(C) = \{i_{V,U} : V \hookrightarrow U \quad | \quad V \subseteq U \subseteq Q(t)\}$$

Where

$i :=$ inclusion of sets ($V \hookrightarrow U := V \subseteq U$)

* Source function ($dom : Mor(C) \to Ob(C)$ such that $dom(i_{V,U} = V)$)

With

dom:= source function

by Math Class ≡ OOP ≡ Logic Structure

Next code about "source class function" can be established (Code in Python on Microsoft Copilot Bing)

Link on matplotlib (source function)

–0–

```
import matplotlib.pyplot

class Ob:
    def __init__(self, Q, subset):
        """
        Q: conjunto base (lista o set de n meros reales)
        subset: subconjunto de Q que queremos considerar como abierto
        """
        self.Q = set(Q)
        self.subset = set(subset)

    def distancia(self, q1, q2):
        """Distancia eucl dea en R"""
        return abs(q1 - q2)

    def es_abierto(self, epsilon=0.1):
```

```
        """
        Verifica si el subconjunto cumple la condici n de abierto:
        para cada punto en el subconjunto, existe un epsilon-ball contenido en e
        """
        for q in self.subset:
            bola = {x for x in self.Q if self.distancia(q, x) < epsilon}
            if not bola.issubset(self.subset):
                return False
        return True

    def __repr__(self):
        return f"Ob({self.subset})"


class Mor:
    def __init__(self, U: Ob, V: Ob):
        """
        Define un morfismo por inclusi n: V     U
        """
        if not V.subset.issubset(U.subset):
            raise ValueError("V no est  incluido en U")
        self.U = U
        self.V = V

    def source_inclusion(self):
        """Devuelve el subconjunto V definido por la inclusi n"""
        return self.V

    def __repr__(self):
        return f"Mor({self.V} -> {self.U})"


# Ejemplo de uso
Q = [x * 0.1 for x in range(-50, 51)]  # Q(t) = puntos entre -5 y 5
U = Ob(Q, [x for x in Q if -2 < x < 2])  # subconjunto abierto
V = Ob(Q, [x for x in Q if -1 < x < 1])  # subconjunto m s peque o

print(" U  es abierto?", U.es_abierto())
print(" V  es abierto?", V.es_abierto())

morfismo = Mor(U, V)
print("Morfismo:", morfismo)
print("Source inclusion:", morfismo.source_inclusion())
```

Result

```
 U  es abierto? False
 V  es abierto? False
Morfismo: Mor(Ob({-0.9, -0.8, -0.7000000000000001, -0.5, -0.6000000000000001, -0
Source inclusion: Ob({-0.9, -0.8, -0.7000000000000001, -0.5, -0.60000000000000001


[Execution Finished]
```

–0–

* Cod function

While the function $cod : Mor(C) \to Ob(C)$ such that $cod(i_{V,U} = U)$)

by Math Class ≡ OOP ≡ Logic Structure

Next code about "cod function" can be established (Code in Python on Microsoft Copilot Bing)

Link on matplotlib (Cod funtion)

–0–

```
import matplotlib.pyplot

class Ob:
    def __init__(self, Q, subset):
        """
        Q: conjunto base (lista o set de n meros reales)
        subset: subconjunto de Q que queremos considerar como abierto
        """
        self.Q = set(Q)
        self.subset = set(subset)

    def distancia(self, q1, q2):
        """Distancia eucl dea en R"""
        return abs(q1 - q2)

    def es_abierto(self, epsilon=0.1):
```

```
        """
        Verifica si el subconjunto cumple la condici n de abierto:
        para cada punto en el subconjunto, existe un epsilon-ball contenido en e
        """
        for q in self.subset:
            bola = {x for x in self.Q if self.distancia(q, x) < epsilon}
            if not bola.issubset(self.subset):
                return False
        return True

    def __repr__(self):
        return f"Ob({self.subset})"


class Mor:
    def __init__(self, U: Ob, V: Ob):
        """
        Define un morfismo por inclusi n: V     U
        """
        if not V.subset.issubset(U.subset):
            raise ValueError("V no est  incluido en U")
        self.U = U
        self.V = V

    def cod(self, obj: Ob):
        """
        Toma un objeto de la clase Ob y regresa el conjunto U
        que est  definido en la inclusi n del morfismo.
        """
        if obj == self.V:
            return self.U
        else:
            raise ValueError("El objeto dado no corresponde al dominio del morfis

    def __repr__(self):
        return f"Mor({self.V} -> {self.U})"


# Ejemplo de uso
Q = [x * 0.1 for x in range(-50, 51)]  # Q(t) = puntos entre -5 y 5
U = Ob(Q, [x for x in Q if -2 < x < 2])  # subconjunto abierto
V = Ob(Q, [x for x in Q if -1 < x < 1])  # subconjunto m s peque o

print(" U  es abierto?", U.es_abierto())
print(" V  es abierto?", V.es_abierto())
```

```
morfismo = Mor(U, V)
print("Morfismo:", morfismo)
print("cod(V):", morfismo.cod(V))
```

$-0-$

Results

```
U es abierto? False
V es abierto? False
Morfismo: Mor(Ob({-0.9, -0.8, -0.7000000000000001, -0.5, -0.6000000000000001, -0
cod(V): Ob({-0.9, -0.8, -0.7000000000000001, -0.6000000000000001, -0.5, -0.4, -0

[Execution Finished]
```

$-o-$

Note that taking the definition of "class of morphism" given early an inclusion morphism $i_{A,D} : A \hookrightarrow D$ can be decomposed if the morphisms

$i_{A,B} : A \hookrightarrow B$

$i_{B,C} : B \hookrightarrow C$

$i_{C,D} : C \hookrightarrow D$

are defined in the class $mor(C)$

For the class $Ob(C)$ that have like elements $U_{Q(t)} \subseteq Q(t)$ such morphisms are defined by the condition of open ball

$d(q(t), x) < r$

Lets remember that

* For each open set $U \subseteq X$, the presheaf F gives a set $F(U)$.

* The elements of $F(U)$ are called **sections on** U.

* These sections represent "local data" defined on U.

Like each $(x, f(x))$ have a dual vector $q(t)$

Taking $I_{P(S)} \subseteq S$

Like

$f(x) := P_x \in I_{P(S)}$

Hence each $f(x)$ have associated a $q(t) \in Q(t)$

Taking

$$c = \{(x_1, x_2) \in \mathbb{C} | -(a+r) \leq x_1 \leq (a+r);$$

$$-\infty < x_2 < \infty;$$

$$a = p\sqrt[n]{\pi^m}\}$$

and the cover (or subcover) defined with $\cup_p B_r(a,b)_p \in \mathbb{C}$ (cover with a+r and subcover with a-r)

where $B_r(a,b)_p$ with $b = 0$

Hence $U_{Q(t)}$ is the set of covectors associated to the set c of $\cup_p B_r(a,b)_p$

With $f(x)$ a function over c of $\cup_p B_r(a,b)_p$

And $F(x)$ one of the interpolator polynomials of the homotopy class defined with $f(x)$ on c of $\cup_p B_r(a,b)_p$

With $S_{F(x)}$:= the set of homogeneous polynomials that defines $F(x)$ on c of $\cup_p B_r(a,b)_p$

Like $U_{Q(t)} \subseteq Q(t)$

Hence exist $I_{U_{F(x)}} \subset S_{F(x)}$ for each $U_{Q(t)} \subseteq Q(t)$

Lets call the elements P_x of this set a section and lets use the etiquette $\mathcal{F}(U_{Q(t)})$;

.i. e. $P_x \in \mathcal{F}(U_{Q(t)})$

with

$\mathcal{F}(S_{F(x)})$:= global sections of $\mathcal{F}$

* Restriction morphism $res_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$
m
Now lets define the function $res_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ – Code on Copilot Microsoft Bing

Code on matplotlib

–0–

```
import matplotlib.pyplot

class Ob:
    def __init__(self, name, subset):
        self.name = name
        self.subset = set(subset)

    def __repr__(self):
        return f"Ob({self.name}, {self.subset})"


class Mor:
    def __init__(self, U: Ob, V: Ob):
        if not V.subset.issubset(U.subset):
            raise ValueError("V no est  incluido en U")
        self.U = U
        self.V = V

    def __repr__(self):
        return f"Mor({self.V.name}      {self.U.name})"


class F:
    def __init__(self):
        self.data = {}

    def asignar(self, obj: Ob, elementos):
        self.data[obj.name] = set(elementos)

    def obtener(self, obj: Ob):
        return self.data.get(obj.name, set())

    def res(self, mor: Mor):
```

```
        """
        Restricci n completa:
        res_V^U : F(U) -> F(V)
        devuelve todos los elementos de F(U) que tambi n est n en F(V)
        """
        F_U = self.obtener(mor.U)
        F_V = self.obtener(mor.V)
        return F_U.intersection(F_V)


# Ejemplo de uso
U = Ob("U", {1, 2, 3, 4})
V = Ob("V", {1, 2})
morfismo = Mor(U, V)

F_obj = F()
F_obj.asignar(U, {"a", "b", "c"})
F_obj.asignar(V, {"a", "b"})

print("F(U):", F_obj.obtener(U))
print("F(V):", F_obj.obtener(V))
print("res_V^U(F(U)):", F_obj.res(morfismo))
```

–0–

Results

```
F(U): {'a', 'b', 'c'}
F(V): {'a', 'b'}
res_V^U(F(U)): {'a', 'b'}


[Execution Finished]
```

* Additional properties

- Identity morphism on $\mathcal{F}(U)$

Using last arguments about res_V^U function "results" must be

$F(U) : \{'a','b'\}$

F(V): {'a', 'b'}

$res_V^U(F(U)) : \{'a','b'\}$

i.e. $res_U^U : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity morphism on $\mathcal{F}(U)$

- Composite $res_W^V \circ res_V^U = res_W^U$

Lets establish

$res_W^V(\mathcal{F}(U) \to \mathcal{F}(V)) : \mathcal{F}(V) \to \mathcal{F}(W)$

Hence whole function is res_W^V

On code this property can be defined as

```
def res_chain(self, mor_list):
        """
        Aplica una cadena de restricciones sucesivas.
        mor_list: lista de morfismos [U->V, V->W, ...]
        """
        current_set = self.obtener(mor_list[0].U)
        for mor in mor_list:
            F_V = self.obtener(mor.V)
            current_set = current_set.intersection(F_V)
        return current_set
```

Or

–0–

```
def res_chain(self, mor_list):
        """
        Restricci n encadenada para cualquier n mero de inclusiones.
        mor_list: lista de morfismos [U->V, V->W, ...]
        """
        if not mor_list:
            return set()

        # empezamos con F(U) del primer morfismo
        current_set = self.obtener(mor_list[0].U)

        # aplicamos intersecciones sucesivas
        for mor in mor_list:
            F_V = self.obtener(mor.V)
            current_set = current_set.intersection(F_V)
```

```
return current_set
```

$-o-$

Taking the general case (see one composition and general case of composition)

Hence next property maintains

* for every three objects a, b, c, a binary operation $\hom(a, b) \times \hom(b, c) \to \hom(a, c)$ called composition of morphisms. Here $\hom(a, b)$ denotes the subclass of morphisms f in $\mathrm{mor}(\mathcal{C})$ such that $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$. Morphisms in this subclass are written $f : a \to b$, and the composite of $f : a \to b$ and $g : b \to c$ is often written as $g \circ f$ or gf.

If the morphism is the identity

$res_W^V \circ res_V^V = res_W^U$

with

$res_W^V(\mathcal{F}(V) \to \mathcal{F}(V)) : \mathcal{F}(V) \to \mathcal{F}(W)$

Hence next property maintains

*the left and right unit laws: for every object x, there exists a morphism $1_x : x \to x$ (some authors write id_x) called the identity morphism for x, such that every morphism $f : a \to x$ satisfies $1_x \circ f = f$, and every morphism $g : x \to b$ satisfies $g \circ 1_x = g$.

If the morphism is

$res_X^W \circ (res_W^V \circ res_V^U) = (res_X^W \circ res_W^V) \circ res_V^U$

Hence associativity can be proved like

Code on Copilot Microsoft Bing (associativity) and Code on Copilot Microsoft Bing (Full Code)

$-0-$

```
def comprobar_asociatividad(self, mor_UV, mor_VW, mor_WX):
    """
```

```
        Verifica que res_X^W     (res_W^V     res_V^U) = (res_X^W     res_W^V)
        """
        # Lado izquierdo: res_X^W(res_W^V(res_V^U(F(U))))
        lado_izq = self.res_chain([mor_UV, mor_VW, mor_WX])

        # Lado derecho: ((res_X^W     res_W^V)     res_V^U)(F(U))
        lado_der = self.res_chain([mor_UV, mor_VW, mor_WX])

        return lado_izq, lado_der, lado_izq == lado_der
```

–0–

And next property maintains

*the associative law: if $f : a \to b$, $g : b \to c$ and $h : c \to d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and

We write $f : a \to b$, and we say f is a morphism from a to b ". We write $\hom(a, b)$ (or $\hom_{\mathcal{C}}(a, b)$ when there may be confusion about to which category $\hom(a, b)$ refers) to denote the hom-class of all morphisms from a to b.

Note that such identity can not be established algebraically and only can be defined like was made on code

Such property can be established through the use of intersection to demonstrate

$$(A \cap B) \cap C = A \cap (B \cap C)$$

* Argumentation about associativity 1
* Argumentation about associativity 2
* Argumentation about associativity 3

Hence the presheaf $\mathcal{F}$ is a category if associativity is considered through such arguments.

* Sheaf of $Q(t)$

* Locality

Lets consider $(x_a, f(x_a)), (x_b, f(x_b)) \in c$ of $\cup_p B_r(x, f(x))_p \in \mathbb{C}$

Like each point have a covector hence exists

$q(t)_a \in U_i \subseteq Q(t)$;

$q(t)_b \in U_j \subseteq Q(t)$

By the presheaf property

$\exists \quad \mathcal{F}(U_i)$ for U_i

and

$\exists \quad \mathcal{F}(U_j)$ for U_j

Like $Q(t)$ is compact $U_i \subseteq Q(t) \subseteq \Re$ is compact if U_i (or j) is closed

hence $\exists$ a cover $\{V_l\}_{l \in L}$ such that $V_l \subseteq U_i$ (or j with index $m \in M$)

By presheaf property $\exists \quad \mathcal{F}(V_l)$ and $\mathcal{F}(V_m)$ for each V_l and V_m

Now lets take

$f(x_a) = s \in \mathcal{F}(U_i)$ and $f(x_b) = t \in \mathcal{F}(U_j)$

and lets apply the $res^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ function for s and t

Hence

$s \in V_l$

and

$t \in V_m$

Hence

If $l = m$

Hence

$s, t \in V_l = V_m$

i.e. $s_{|_{V_l}}, t_{|_{V_m}}$

If $s_{|_{V_l}} = t_{|_{V_m}}$

for all $i, j \in L$ or M

hence

$s = t$ (i.e. $f(x_a) = f(x_b)$)

* Gluing

Lets consider one of the interpolator polynomial of the class of homotopy ; $F(x'_i) = y_i$ or $G(x'_j) = y_j$ such that $x_i = \gamma(y_i)$ for $x_i \in [0,1]$ and $x' \in Dom(f(x))$

and lets take $(x'_i, F(x'_i))$, $(x'_j, G(x'_j)) \in c$ of $\cup_p B_r(x'_i, f(x_i))_p$

such that $F(x'_i), G(x'_j) \in Im(f(x_i))$

Like each pair have a covector hence

and

$q(t)_i \in U_i \subseteq Q(t)$

$q(t)_j \in U_j \subseteq Q(t)$

Like $Q(t)$ is compact hence $\exists$ $\{V_l\}_{l \in L}$ for U_i (or j with index $m \in M$) if is closed

i.e. $V_l \subseteq U_i$ (or m for j)

By presheaf property hence

$\exists$ $\mathcal{F}(U_i)$ for each U_i and $\mathcal{F}(V_l)$ for each V_l (or j with index $m \in M$)

Now consider $F(x'_i) = s_l \in \{\mathcal{F}(V_l)\}_{l \in L}$ (or j and M)

Besides consider $res^U_{V_l \cap V_{l'}} : \mathcal{F}(U_i) \to \mathcal{F}(V_l \cap V_{l'})$

for $s_l \in V_l$ and $s_{l'} \in V_{l'}$ (or M and j)

If $s_{l|_{V_l \cap V_{l'}}} = s_{l'|_{V_l \cap V_{l'}}}$

for all l, l' in L (or m, m' in M)

Like res^U_V establish $V_l \cap V_{l'} \subseteq U_i$ (or $m, m' \in M$ for j)

Hence $\exists s_l$ (or l') $\in \mathcal{F}(U_i)$ or (U_j)

such that

$res_V^U : \mathcal{F}(U_i) \to V_l$ (or j and m) for $s_l \in \mathcal{F}(U_i)$

like $s_{l_{|V_l \cap V_{l'}}} = s_{l'_{|V_l \cap V_{l'}}}$

Hence $s_{|V_l} = s_{|V_l \cap V_{l'}}$

Note that if inverse res is defined for $V_l \cap V_{l'} = U_i$ i.e. $U_i \subseteq V_l \cap V_{l'}$ the postulate is also true.

Hence the presheaf if also a sheaf

$$-0-$$

* Ringed space of $Q(t)$ of homogeneous polynomials P_k

Now lets take the homogeneous polynomials $s := F(x')$ of the sheaf that form a ring and its covectors i.e.

$$O_{P_k(x)} = \{ \quad q(t) \quad | \quad q(t) \quad \text{is the covector of} \quad (x', f(x')) \in \mathbb{C}$$

$$q(t) \in U \subseteq Q(t)$$

$$\text{with } f(x') = F(x') = y := P_k \in \mathcal{F}(U)$$

$$\text{with } x = \gamma(y) \text{ for } x \in [0, 1]$$

$$\text{with } F(x') := \text{ homogeneous polynomial } \}$$

Like $Q(t)$ is a topological space hence

$(Q(t), O_{P_k(x)}) :=$ ringed space

or

$O_{P_k(x)}$:= the ringed space of S := the ring of homogeneous polynomials that defines $F(x)$:= the interpolator polynomial

i.e. $O_{P_k(x)}$:= structure sheaf of $Q(t)$ of homogeneous polynomials P_k

* Stalk of the sheaf of $Q(t)$ of homogeneous polynomials P_x

Def. Before In this section objects are understood to consist of underlying sets equipped with a given algebraic structure, such as groups, rings, modules (over a fixed ring), algebras (over a fixed field), etc. With this in mind, homomorphisms are understood in the corresponding setting (group homomorphisms, etc.).

Let $\langle I, \leq \rangle$ be a directed set. Let $\{A_i : i \in I\}$ be a family of objects indexed by I and $f_{ij} \colon A_i \to A_j$ be a homomorphism for all $i \leq j$ with the following properties:

1. f_{ii} is the identity on A_i, and

2. $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$.

Then the pair $\langle A_i, f_{ij} \rangle$ is called a direct system over I.

The direct limit of the direct system $\langle A_i, f_{ij} \rangle$ is denoted by $\varinjlim A_i$ and is defined as follows. Its underlying set is the disjoint union of the A_i's modulo a certain equivalence relation $\sim$:

$$\varinjlim A_i = \bigsqcup_i A_i \Big/ \sim$$

Here, if $x_i \in A_i$ and $x_j \in A_j$, then $x_i \sim x_j$ if and only if there is some $k \in I$ with $i \leq k$ and $j \leq k$ such that $f_{ik}(x_i) = f_{jk}(x_j)$. Intuitively, two elements in the disjoint union are equivalent if and only if they "eventually become equal" in the direct system.

Def. Now. In this section objects are understood to consist of underlying sets equipped with a given algebraic structure, such as groups, rings, modules (over a fixed ring), algebras (over a fixed field), etc. With this in mind, homomorphisms are understood in the corresponding setting (group homomorphisms, etc.).

Let $\langle I, \leq \rangle$ be a directed set. Let $\{A_i\}_{i \in I}$ be a family of objects indexed by I and $f_{ij} \colon A_i \to A_j$ be a morphism for all $i, j \in I$ with the following properties:

1. f_{ii} is the identity on A_i, and

2. $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$.

Then the pair $\langle A_i, f_{ij} \rangle$ is called a direct system over I.

The direct limit of the direct system $\langle A_i, f_{ij} \rangle$ is denoted by $\varinjlim A_i$ and is defined as follows. Its underlying set is the disjoint union of the A_i's modulo a certain equivalence relation $\sim$:

$$\varinjlim A_i = \bigsqcup_i A_i \Big/ \sim$$

Here, if $x_i \in A_i$ and $x_j \in A_j$, then $x_i \sim x_j$ if and only if there is some $k \in I$ with $i \leq k$ and $j \leq k$ such that $f_{ik}(x_i) = f_{jk}(x_j)$. Intuitively, two elements in the disjoint union are equivalent if and only if they "eventually become equal" in the direct system.

Def. In mathematics, a directed set (or a directed preorder or a filtered set) is a preordered set in which every finite subset has an upper bound. In other words, it is a non-empty preordered set A such that for any a and b in A there exists c in A with $a \leq c$ and $b \leq c$.

Def. A binary relation $\lesssim$ on a set X is called a preorder or quasiorder if it is reflexive and transitive; that is, if it satisfies:

1. Reflexivity: $a \lesssim a$ for all $a \in X$ and

2. Transitivity: if $a \lesssim b$ and $b \lesssim c$ then $a \lesssim c$ for all $a, b, c \in X$.

A set that is equipped with a preorder is called a preordered set (or proset)

Proof.

Lets take

$$c = \{(x_1, x_2) \in \mathbb{C} | -(a + r) \leq x_1 \leq a + r;$$

$$-\infty < x_2 < \infty;$$

$$a = p \sqrt[n]{\pi^m};$$

$$\}$$

now lets define c_p where $p \in [1, 2, 3, ...]$

such that

c_1 is defined with $p \in [1]$

c_2 is defined with $p \in [1, 2]$

c_3 is defined with $p \in [1, 2, 3]$

$\vdots$

Where each set have cover in $\cup_p B_r(a, b)_p$

Now lets consider that f(x) is defined in c_p and lets take the points of homology defined with $F(x)$ and $G(x)$ with $\gamma(f(x')) = x$ (where $x \in [0, 1]$) and lets consider the points $(x', f(x')) \in B_r(x', f(x')) \subset c_p$

with

$$B_r(x', f(x')) = \{ \quad (x_1, f(x_1)) \in \mathbb{C} \quad | \quad d[(x', f(x')), (x_1, f(x_1))] < r \quad \}$$

Like each $(x', f(x'))$ have a covector hence the set of $q(t)$ defined on c_p is

$$q(t) \in U_p \subset Q(t)$$

Now remember that by sheaf property if U_p is closed $Q(t)$ is compact and hence $\exists$ a cover $\{V_l\}_{l \in L}$ for U_p such that

$$V_l \subseteq U_p$$

Now lets define $A_i := V_l$

Like $\{V_l\}_{l \in L}$ is a cover and $l \in L$ is an index hence $\{V_l\}_{l \in L}$ is a family of objects indexed by L

Like $L := N$ and taking $\leq$ like binary relation; hence L is a preorder because $l = l$ (reflexive) and for l_a, l_b, l_c ; with $l_a \leq l_b$ and $l_b \leq l_c$ hence $l_a \leq l_c$

In particular $\leq$ is also antisymmetric; i.e. if $l_a \leq l_b$ and $l_b \leq l_a$ hence $l_a = l_b$; hence $\leq$ is also a partial order

and like every pair of l_a and l_b is comparable (i.e. $l_a \leq l_b$ or $l_b \leq l_a$) hence $\leq$ is a total order

Now lets take $U_L \subseteq L$ a subset on L like $\leq$ is an order relation on L hence $\exists n \in N$ such that $l \leq n; \quad \forall l \in U_L$

Hence U_L have upper bound and

Hence L with $\leq$ define a directed set

Now lets define $f_{ij} : A_i \to A_j$

Like $res_V^U := \mathcal{F}(U) \to \mathcal{F}(V)$ is a morphism

with $res_V^V :=$ identity

f_{ii} identity on A_i property is fulfilled

while with

$$res_V^W \circ res_V^U = res_W^U$$

$f_{ik} = f_{ij} \circ f_{jk}$ property is fulfilled

Hence the pair V_l and res_V^U is a direct system over L

* Direct limit $(lim_{\to} V_l)$

Now lets define disjoint union of $V_l s$ modulo $\sim := s_{|_{V_l}} = s_{|_{V_l'}}$

Def. An equivalence (or congruence) relation R, a is equivalent (or congruent) to b modulo R if aRb.

Now lets take U_i and U_j closed hence by sheaf property $\exists \quad \{V_l\}_{l \in L}; \{V_m\}_{m \in M}$ a cover of U_p and $U_p' \in Q(t)$

By presheaf property $\exists \quad \mathcal{F}(U_p)$ and $\mathcal{F}(V_l)$ (or p' and m respectively)

Now applying res_V^U morphism to each presheaf $\mathcal{F}$

$res_V^U : \mathcal{F}(U_p) \to \mathcal{F}(V_l)$;

$res_V^U : \mathcal{F}(U'_p) \to \mathcal{F}(V_m)$

with

$s \in \mathcal{F}(U_p)$ hence $s_{V_l} \in \mathcal{F}(V_l)$

and

$s' \in \mathcal{F}(U'_p)$ hence $s'_{V_m} \in \mathcal{F}(V_m)$

Like $\sim$ establish that $s_{V_l} \sim s'_{V_m}$ if $V_l \cap V_m \neq \phi$

* If $U_p \cap U'_p = \phi$ hence $V_l \cap V_m = \phi$

* While if $s, s' \in \mathcal{F}(U_p)$ (or $\mathcal{F}(U'_p)$)

with V_l and $V'_l \in \{V_l\}_{l \in L}$

and applying res_V^U morphism to the inclusion $V_l \subseteq U_p$ and $V'_l \subseteq U_p$ (or U'_p)

Hence $V_l \cap V'_l \neq \phi$

If $V_l = V'_l$ hence $\sim$ is reflexive because $s_{|_{V_l}} = s_{|_{V_l}}$

*Now taking $V_l, V'_l \subseteq U_p$ (or U'_p by $V_l \in \{V_l\}_{l \in L}$) such that $V_l \neq V'_l$

Hence $V_l \cap V'_l \neq \phi$ while by commutativity of intersection by pairs $V'_l \cap V_l \neq \phi$

Hence $\sim$ is symmetric

* Taking $V_l, V'_l, V''_l \in U_p$ by $V_l \in \{V_l\}_{l \in L}$

If $V_l \cap V'_l \neq \phi$

$V'_l \cap V''_l \neq \phi$

hence

$s_{V_l} \sim s_{V'_l}$

and

$s_{V'_l} \sim s_{V''_l}$

If

$V \subseteq V'$

or

$V'' \subseteq V$

Hence

$\sim$ is transitive

Hence

$\sim$ is an equivalence relation

Hence

$\mathcal{F}_x := lim_{\to U_p \ni q(x)} \mathcal{F}(U_p) := \sqcup_l \{V_l\}_{l \in L} / \sim$

i.e. the disjoint union modulo $\sim$ is the stalk of a sheaf $\mathcal{F}$ at $q_x \in V_l$

* Locally ringed space of homogeneous polynomials P_k

Proof.

Lets take the ringed space $O_{P_{k(x)}}$ on U_p such that S_{U_p} := the set of the homogeneous polynomials $F(x')$ of the sheaf that form a ring and its covectors; i.e.

$$O_{P_k(x)} = \{q(t) \quad | \quad \text{is the covector} \quad (x', f(x')) \in \mathbb{C};$$

$$q(t) \in U_p \subseteq Q(t);$$

$$\text{with} f(x') = F(x') = y := P_k \in \mathcal{F}(U_p);$$

$$\text{with } x = \gamma(y) \text{ for } x \in [0, 1];$$

$$\text{with } F(x') := \text{ homogeneous polynomial}$$

$$\}$$

with $Q(t)$ a topological space

such that $P_k := [f_{U_p}] \in \sqcup_l V_l / \sim$

with S_{U_p} have a unique maximal ideal I_{U_p} i.e. all the points of S_{U_p} fulfilled next properties

$P_k + P'_k \in I_{S_{U_p}}$

$-P_k \in I_{S_{U_p}}$

$rP_k \in I_{S_{U_p}}$

for a function $f(x')$ defined on c_p

if the points P_k of the interpolator polynomial that define $f(x')$ on $c_{p'}$ not fulfill one or more of the last conditions about the ideal $I_{S_{U_p}}$ hence

$\exists \quad U_p \in O_{P_k}$ s.t. $\mathcal{F}(U_p)$ not a unique maximal ideal in O_{P_k}; i.e. $\mathcal{F}(U_P)$ is not a local ring

Hence S_{U_p} is a locally ringed space

i.e.

$$S_{U_P} = \{ \quad F(x') \in \mathcal{F}(U_p) \quad | \quad \mathcal{F}(x) \text{ is a sheaf ;}$$

$$\text{where } q(t) \in O_{P_k(x)};$$

$$f(x) \text{ is defined on } c_p;$$

$$P_k \in \sqcup_l V / \sim$$

$$\exists \quad U_p \text{ s.t. } O_{P_k} \text{ is not a local ring ;}$$

i.e. is not a unique maximal prime ideal

}

* Affine scheme of S_{U_p}

Remember that the prime spectrum of a commutative ring is

$$Spec(S) = \{ \quad I_{P(S)} \subseteq S \quad | \quad I_{P(S)} \text{ is a prime ideal} \quad \}$$

else with S_{U_P} defined as early on last item

With $P_k \in I_{P(S)}$

and $P_k := F(x') \in S_{U_p}$

Lets take

$$z(x_1, x_2) := g[Prod(x_1, x_2)] = Prod(g(x_1, x_2))$$

Like z is an homomorphism an is bijective hence $z(x_1, x_2)$ is an isomorphism

Like every maximal ideal is prime hence

If $I_{S_{U_p}} \in Spec(S)$

Hence

S_{U_P} is an affine scheme

* Scheme of S_{U_p}

Lets take $U_p \subset Q(t)$

If U_p is closed hence $\exists \{V_l\}_{l \in L}$

s.t. the family of open sets is a cover of U_p by sheaf property

by presheaf property $V_l \subset U_p$

and $\mathcal{F}(U_p) \to \mathcal{F}(V_l)$

with $s := F(x') \in \mathcal{F}(U_p)$

By ringed property $O_{P_k(x)}$ is the set of the covectors of the pairs $(x', F(x'))$ such that $F(x')$ is an homogeneous polynomial where $q(t) \in O_{P_k(x)}$ form a ring

with $[F(x')] \in \sqcup_l \{V_l\}_{l \in L} / \sim$ by stalk property

Like $\exists I_{S_{U_p}}$ for every $\sqcup_l \{V_l\}_{l \in L} / \sim$ (every stalk) with $[F(x')]_{I_{S_{U_p}}} \in I_{S_{U_p}}$ by locally ringed space

i.e. $I_{S_{U_p}} \subseteq S_{U_p}$

Like $U_p \subseteq Q(t)$ is compact and have an ideal $I_{U_p} \subseteq U_p \subseteq Q(t)$

i.e. $\exists$ a cover $\{I_m\}_{m \in M}$ for I_{U_p} where m is an index defined with

$$p_m = \frac{a}{m}$$

$$q(t) = (\frac{n}{q_{e^-}}) \frac{h\nu}{\frac{1}{2} m_{e^-} (\frac{dx}{dt})^2}$$

with $dt = t' = \frac{p_m dx}{\lambda \nu}$

or simplified by

$q(t) = \frac{a}{(\frac{dx}{dt})^2}$

by presheaf property

with $\mathcal{F}(I_{U_p}) \to \mathcal{F}(I_m)$

Hence I_{U_p} is a sheaf and $\mathcal{F}(I_m)$ is an indexable set

Like $s_{I_m} \in \mathcal{F}(I_m) \subset \mathcal{F}(I_{U_p})$

If $\cup_{m \in M} \mathcal{F}(I_m) = \mathcal{F}(U_p)$

Hence $\exists$ a cover $\{\mathcal{F}(I_m)\}_{m \in M}$ of $\mathcal{F}(U_p)$

with

$$\mathcal{F}(I_m) = \{ \quad P_{k_{I_m}} \quad | \quad d(P_{k_{I_m}}, P_{k_x}) < r_{I_m} \}$$

an open set

and

$$\mathcal{F}(I_{U_p}) = \{P_{k_{U_p}} \quad | \quad d(P_{k_{U_p}}, P_{k_x}) < r_{U_p} \}$$

hence

$$d(P_{k_{I_m}}, P_{k_x}) < d(P_{k_{U_p}}, P_{k_x})$$

$\forall P_{k_{I_m}} \in I_{U_p}$

with

$$\mathcal{F}(I_m) := S_{I_m}$$

the set of homogeneous polynomials of the ideal of $q(t) \in I_m$

Like every maximal ideal is a prime

and by stalk property about $I_{S_{U_p}}$ i.e.

$$[P_k] \in \{I_m\}_{m \in M} / \sim$$

by locally ringed space property

$\exists I_{I_m}$ and ideal of $P_k \in [P_k] \in \{I_m\}_{m \in M} / \sim$ such that

if $\forall P_k \in I_{I_m}$

$P_{k1} + P_{k2} \in I_{I_m}$

$-P_k \in I_{I_m}$

$rP_k \in I_{I_m}$

Like $\{I_m\}_{m\in M}$ is a cover of I_{U_p}

and if

$$\cup_{m\in M}\mathcal{F}(I_m) = \mathcal{F}(I_{U_p})$$

Hence if

$\cup_{m'\in M'}\mathcal{F}(I'_m) = I_{I_m}$

with $M' < M$

and

$$\cup_{m'\in M'}\mathcal{F}(I'_m) \subseteq \cup_{m\in M}\mathcal{F}(I_m)$$

Hence exist a cover of I_{I_m} such that

If $I_{I_m} \in Spec(S)$

hence

Every I_{I_m} is isomorphic to $Spec(S)$

i.e. is an affine scheme

Hence $S_{I_{U_p}} := \mathcal{F}(I_{U_p})$ is an scheme

* Projetivization of V

Proof. First considerate next definitions

Def. Let S be a commutative graded ring, where

$$S = \oplus_{i\geq 0} S_i$$

is the direct sum decomposition associated with the gradation. The irrelevant ideal of S is the ideal of elements of positive degree

$$S_+ = \oplus_{i>0} S_i$$

An ideal is homogeneous if it is generated by homogeneous elements.

Then, as a set,

$$ProjS = \{P \subseteq S \text{ homogeneous prime ideal, } S_+ \nsubseteq P\}$$

$$-0-$$

Remember that

$$S = \{P_k \in K[x_0, ..., x_n] \quad | \quad I_{P(S)} \text{ is a prime ideal with } P_k \text{ a monomoium } \}$$

Now lets grade S

$$-0-$$

Def. A graded ring is a ring that is decomposed into a direct sum

$$R = \bigoplus_{n=0}^{\infty} R_n = R_0 \oplus R_1 \oplus R_2 \oplus ...$$

of additive groups, such that

$$R_m R_n \subseteq R_{m+n}$$

$$-0-$$

First lets prove that S is a base for $k[V]$ the space of polynomials

$$-0-$$

Def. A basis B of a vector space V over a field F is a linearly independent subset of V that spans V. This means that a subset B of V is a basis if it satisfies the two following conditions:

* Linear independence: For every finite subset $\{v_1, ..., v_m\}$ of B, if $c_1v_1 + ... + c_mv_m = 0$ for some $c_1, ..., c_m$ in F, then $c_1 = ...c_m = 0$; and

* the spanning property: for every vector v in V, one can choose $a_1, ..., a_n$ in F and $v_1, ..., v_n$ in B such that $v = a_1v_1 + ... + a_nv_n$. In other words, v can be represented as a linear combination of some vectors in B.

$-0-$

*

Lets take $P_{ki} \in S$ that defines interpolator polynomial of class homology and $a_i \in F$ (a field) such that

$a_1P_{k1} + a_2P_{k2} + a_3P_{k3} + ... + a_nP_{kn} = 0$

Like $P_{ki} \neq 0$ hence

$c_i = 0$ hence S is linearly independent

*

Now lets consider $a_i \in F$ and $P_{ki} \in S$

such that

$v = a_1P_{k1} + ... + a_nP_{kn} \in V$

Hence $V = span[S]$

hence S is a base for V

*

Like $v = a_1P_{k1} + ... + a_nP_{kn} \in V$

Hence

$V = \bigoplus_{k\geq 0} V_k$

with

$$V_k = \{aP_k \quad | \quad k := \text{ grade of the power and } P_k := b_i x^k \text{ with } a_i \in F\}$$

with $v \in \bigoplus_{k \geq 0} V_k = V$

Now lets consider that

$P_i \in V_i$

and

$P_j \in V_j$

$abx^i \cdot a'b'x^j = aa'bb'x^{i+j} \in V_{i+j}$

hence

$V_i V_j \subseteq V_{i+j}$ by direct sum property and V_k definition

Hence like addition and multiplication of polynomials in the $span[S] = V$ is defined with $p_k = bx^k$ and $v = a_i P_k$

hence V is a ring

Like V is a ring and a vector space

hence V is an algebra over F

* Irrelevant ideal of V

Now lets consider the set $I_{V_+} = \bigoplus_{i>0} V_i$ of the elements $P_k \in V$ that fulfill next requirements

$P_{k1} + P_{k2} \in I_{V_+}$

$-P_k \in I_{V_+}$

$rP_k \in I_{V_+}$

Hence I_{V_+} := an ideal of elements of positive degree

Lets define the spectrum of V

Like V is commutative hence is a commutative ring

Now lets consider its power set $P(V)$

Now lets consider the ideals ring prime $I_V \subset V$ such that $I_V \subseteq P(V)$

Hence

$$Spec(V) = \{I_V \subseteq V \quad | \quad I_V \text{ is a prime ideal}\}$$

is the spectrum of the ring V

and lets define

$$ProjV = \{I_V \subseteq V \text{ homogeneous prime ideal, } I_{V_+} \nsubseteq I_V\}$$

Now lets consider $\mathbb{C}$ and its dual vector $Q(t)$

Like each $(x, f(x)) \in \mathbb{C}$ have a covector $q(t) \in Q(t)$

and like $q(t) \in U \subset Q(t)$ have a section $s \in \mathcal{F}(U)$

with $s := F(x) = f(x)$

an homogeneous polynomial (a monomial $P_k = ax^n$)

by presheaf property

Hence taking the stalk of the sheaf of the presheaf defined

The equivalence class of $[f(x)] := [s]$ is defined

By locally ringed space all stalks of O_{P_k} (the ringed space $q(t) \in Q(t)$) have an ideal I_{I_m} of sections i.e.

$s \in I_{I_m}$

Like each $U \subseteq Q(t)$ have a cover hence the ideal I_{I_m} have a cover by sheaf property

Like each ideal I_{I_m} of the equivalence class of each stalk of each U_i is isomorphic to the spectrum of S (the set of monomials or homogeneous polynomials) hence is an affine scheme and I_{I_m} an scheme

Taking the span of I_{I_m} the vector space of polynomials V can be defined

$$span[I_{I_m}] = V$$

Like $v = a_1 P_{k1} + ... + a_n P_{kn}$

and two bases can generate the same vector space

Hence $span[I_{I_m}] = span[S]$

Hence

$span[I_{I_m}]$ is a graded ring

with

$I_{span[I_{I_{m+}}]}$ an ideal of elements of positive degree of V

Hence

$$ProjV = \{I_{I_m} \subseteq V \text{ homogeneous prime ideal, } I_{span[I_{I_{m+}}]} \nsubseteq I_V\}$$

The scheme $S_{I_{U_p}} := \mathcal{F}(I_{U_p})$ over $Q(t) = V^*$ defined by $Proj[V]$ is called projectivization of V

** Projective Variety

Lets define the points of P_k

Lets define K the set of the points

$$K = \{x \in \Re \quad | \quad P_k := f(x) \text{ is a homogeneous monomial and } k := \text{ degree } \}$$

Lets consider that early

$x \in [x-\delta, x+\delta]$

with $I = [x-\delta, x+\delta]$

$|I| = (x+\delta) - (x-\delta) = 2\delta$

$P(I) = \{E_i | E_i \subseteq [x-\delta, x+\delta]\}$

where $P(I) :=$ is the power set of I

i.e. $E_i \subseteq I$

Remember that on a measure space $\mu(E_i) = 1$

If $\sum_k P_k = 1$ i.e. the sum of the homogeneous polynomials is equal 1 hence if P_k have domain over E_i hence the polynomial is a probability measure.

like

E/E' is a line on the quotient group hence for $\Re$ is a segment on $\Re$ by the length of each interval (a vector on $\Re$)

If $\mu(E/E') = 1$ hence μ is a probability measure

Hence the set of points

$Dom(P_k) = \{x_i \in \Re | P_{k1}(x_1) + P_{k2} + ... = \mu(E_i) \text{with} \mu(E_i) = 1 \text{a mesure probability and} x_i \in (x_i - \delta, x_i + \delta)\}$

Note that under this assumption if $\mu(E_i) = 0$ is defined hence the set

$Null(P_k) \subseteq Dom(P_k) = \{x_i \in \Re | P_{k1}(x_1) + P_{k2} + ... = 0 = \mu(E_i) \text{ with } \mu(E_i) = 1 \text{ a measure probability and} x_i \in (x_i - \delta, x_i + \delta)\}$

i.e. exist the zero locus of S by $x \in Dom(s)$ with $s \in S_{I_{U_p}}$

taking

$$ProjV = \{I_{I_m} \subseteq V \text{ homogeneous prime ideal, } I_{span[I_{I_{m+}}]} \nsubseteq I_V\}$$

The scheme $S_{I_{U_p}} := \mathcal{F}(I_{U_p})$ over $Q(t) = V^*$ defined by $Proj[V]$ is called projectivization of V

A subset S_V of $S_{i_{U_p}}$ where $\sum_k P_k = F(x)$ is defined and $F(x) = 0$ is equal to the zero locus $Z(S)$ if $x \in Dom(s)$ with $s \in S_{I_{U_p}}$

i.e.

$S_V = Z(S)$

- Irreducibility S_V

with $E_i \cap E_j \neq 0$ by Axiom union choice

Like μ is defined for each E_i i.e. each set have a domain of μ hence such E_i define the probability measure also consider that with $x \in E_i \cap E_j = 0$ by projectivization property

Like $\mu(E_i \cap E_j) \neq \mu(E_i)$ and $\mu(E_j)$ for axiom intersection choice

and

$\mu(E_i \cap E_j) = \mu(E_i)$ within $\mu(E_j)$ for projectivization

where $\mu(E_i) + \mu(E_j) = 1$

hence E_i is irreducible to define μ the probability measure when E_i´s follow Axiom intersection choice

Hence S_V is an irreducible projective variety i.e. a projective variety

-o-

With Kähler manifold

The algebraic variety structure defined through pre sheaf category $\mathcal{F}(x)$ establish homotopy polynomial (interpolator) class of the holonomy group property to define a section s and a monomial P_k

$Hom_(f_i) = \{f_i | f_i$ is homotopic tof_js.t.$x_i = \gamma_1(\gamma_i)$ and $x_j = \gamma_2(y_j)\}$

With $E_x := \mathbb{C}$

and the holonomy set of the elements of its group

$Hol_x(\nabla) = \{x_{GL} \in GL(E_x) | e_{\Re}(x_i, f(x_i)) \in CSB$ and $x_i = \gamma(f(x_i)) \neq \gamma(f(x_j)) = x_j\} = \{e_{\Re}\}$

With the connection defined with $(x_1, f(x_1)), (x_2, f(x_2)) \in T_{p_T}M$ and $detA = 0$ the lines are parallel (for $\Gamma = \frac{1}{2}\frac{1}{detA}(\delta_i g_{ii})$) and old condition $\nabla : \Gamma(E) \rightarrow \Gamma(T^*M \otimes E)$

and

$\nabla : T_{p_T}M \rightarrow (E_{\|}/N_{E_{f_{\|}}})^* \otimes (T_{p_{f_{\|}}}M/N_{T_{p_{f_{\|}}}M})^*$

A section of the fiber $\sigma_{f_{\|}} : CSB_{f_{\|}} \rightarrow E_{f_{\|}}$ is parallel if $f_{\|}(x)$ to f(x) is parallel

Hence the connection ∇ defines a parallel transport map from $T_{p_T}M$ to $E_{f_{\|}}$ on the fiber of E at $(x_{f_{\|}}, f_{\|}(x_{f_{\|}}))$

Hence the parallel transport is defined on

$P_x : T_pM \rightarrow E(or E_x)$

i.e. the set $Dom(P_x) := T_pM$

(i.e.

Like E the Riemann space tangent to T_pM is also a symplectic and a complex manifold

Hence the tangent space to the Riemann manifold (with Riemannian metric); i.e. T_pM is also tangent to a symplectic and complex manifold E

Hence the elements $(x, f(x))$ that form the algebraic variety are on $\sigma_{f_{\|}}$ a section of the fiber bundle on a Riemann manifold (a symplectic and complex)

Hence the elements of the algebraic variety are on a Kähler Manifold

Note that if $F(x, y)$ exists hence with θ a direction to define each point; such vector function define whole space; hence the algebraic variety form (construct) a Kähler manifold. (If the interpolator polynomial is considered as the congruence of triangles; defined for each direction of each point of the vector space where the vector function is defined like a random variable).

18 Hodge Conjecture

Remember that

Def. Harmonic Form. By definition a form Ω^k (or k $\omega-$ form) on M is harmonic if its Laplacian is zero:

$$\mathcal{H}^k_{\triangle} = \{\alpha \in \Omega^k(M) \quad | \quad \triangle\alpha = 0\}$$

Remember that the $2-\omega$ form is

the smooth section

$\sigma : CSB_{(h_1,h_2)} \to E$

of the 2 - exterior power of the cotangent bundle

$$\alpha \wedge \beta := \begin{cases} (a,b) \in X \times X \subseteq CSB_{(h_1,h_2)} & ; \\ ae_i + be_j = c \in R = T^2(X) & ; \\ c_1 - c_2 \in I \text{ such that } c = xr(rx) & \end{cases}$$

Def. For a complex manifold X and a natural number r, every C^∞ r-forms on X (with complex coefficients) can be written as a sum of forms of type (p,q) with $p+q=r$, meaning forms that can locally be written as a finite sum of terms, with each term taking the form

$$f dz_1 \wedge \cdots \wedge dz_p \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_q$$

with f a C^∞ function and the z_s and w_s holomorphic functions.

Now lets consider the holomorphic properties of z_s and w_s

Def. (Before) $f(z)$ a complex function can be decomposed into u and v; such that $f(z) = f(x,y) = f(x+iy) = u(x,y) + iv(x,y)$

Hyp.

$f(z) := f(x,y) := g[Add(x,y)] = g(x+y) = g(x) + g(y)$

$= g_u(x) + g_u(y) + g_v(x) + g_v(y)$

$= g_u(x+y) + g_v(x+y) = g_u(Add(x,y)) + g_v(Add(x,y))$

$= u(x,y) + ig_v(x,y)$

Such that

Hyp. (Complex Function Derivability) $g_u(x) = g_v(y)$ and $g_u(y) = -g_v(x)$

and

$$g(x) = g_u(x) + ig_v(x) \quad ;$$

$$g(y) = g_u(y) + ig_v(y)$$

with

$$g_v(x) := g_u(x + h)$$

hence quotient by h is

$$\frac{dg}{dx} = lim_{h \to 0} \frac{g_u(x + h) + g_u(x)}{h}$$

derivative (and the same for $g_u(y)$)

Def. A function is **holomorphic** on an open set U if it is *complex differentiable* at *every* point of U. A function f is *holomorphic* at a point z_0 if it is holomorphic on some neighbourhood of z_0

Now lets suppose

$$fdz = g_u(x) + g_u(x + h)$$

clearing dz

$$dz = \frac{g_u(x) + g_u(x + h)}{f}$$

integrating

$$z = \int \frac{g_u(x) + g_u(x + h)}{f} dx$$

like f is a coefficient function hence f is $f(x)$

while

$d\bar{w} = g_u(y) + g_u(y+h)$

integrating

$w = \int [g_u(y) + g_u(y+h)]dy$

Lets take g_u and g_v holomorphic ; i.e. the limit to h is defined for both g_u and g_v functions; besides also lets take last condition of integration

Now lets obtain its Laplacian

With $a := fdz$ and $b := d\bar{w}$

Lets take

$$(fdz, d\bar{w}) \in \mathbb{C}$$

Now lets establish the quotient by h

$$(\frac{-g_u(x) + g_u(x+h)}{h}; \frac{-g_u(y) + g_u(y+h)}{h})$$

and lets obtain its limit (first derivative)

$$(lim_{h\to 0}\frac{-g_u(x) + g_u(x+h)}{h}; lim_{h\to 0}\frac{-g_u(y) + g_u(y+h)}{h})$$

Now lets apply Add function

$$Add(lim_{h\to 0}\frac{-g_u(x) + g_u(x+h)}{h}; lim_{h\to 0}\frac{-g_u(y) + g_u(y+h)}{h})$$

$$= lim_{h\to 0}\frac{-g_u(x) + g_u(x+h)}{h} + lim_{h\to 0}\frac{-g_u(y) + g_u(y+h)}{h}$$

with $y = f(x)$

$$= lim_{h\to 0}\frac{-g_u(x)+g_u(x+h)}{h} + lim_{h\to 0}\frac{-g_u(f(x))+g_u(f(x)+h)}{h}$$

by definition

$$= \frac{da}{dx} + \frac{db}{dx}$$

Now lets obtain the derivative

$$= \frac{d}{dx}(\frac{da}{dx} + \frac{db}{dx})$$

$$= \frac{d^2a}{dx^2} + \frac{d^2b}{dx^2}$$

that is by definition the Laplacian

Lets note that this term is zero if

$$\frac{d^2a}{dx^2} = \frac{d^2b}{dx^2}$$

or if the coefficient $f(x) = 0$

Like a and b are twice differentiable hence are harmonic functions

Like the Laplacian of ω on $\mathbb{C}$ is zero hence its form is harmonic

Lets take the algebraic variety of the Kähler manifold; by presheaf property the inclusion

$V \subset U$ is defined for all

With $f(x) = P_x \in \mathcal{F}(V)$

Remember that the immersion $f : N \to M$ establish that its pullback $f^*_{pT}\omega : T_{p_T}N \to T_{f(p_T)}M$ is collinear and hence $N \subset M$ (integrable condition) with

x and $y = f(x)$ used to define $u := g_u$ and $v := g_v$

and $(x, f(x)); (u, v) \in \mathbb{C}$

with

N a submanifold of $M = \mathbb{C}$

and $(x, f(x)) \in N \subset M \subseteq \mathbb{C}$

The cohomology group of N is

$$H_{d_{\leftarrow}-N} = \frac{ker(d_{\leftarrow})_N}{Im(d_{\leftarrow})_N}$$

with lateral class

$$ker(d_{\leftarrow})_N / Im(d_{\leftarrow})_N = \{x_{ker} \cdot Im(d_{\leftarrow})_N \quad | \quad x_{ker} \in ker(d_{\leftarrow})_N\}$$

$$d_{\leftarrow} := \begin{cases} x \in \bigcap_{i=1}^{\infty} \{E_i\}_{i=0}^{\infty} & ; \\ E_0 \supseteq E_1 \supseteq E_2 \supseteq ... & ; \\ E_0 = [x - \delta, x + \delta] & ; \\ \delta_i = \delta_0(\#steep(i)) & ; \\ P_1(x) = g(x - \delta) + \frac{g(x+\delta) - g(x-\delta)}{2} & ; \\ P_1(x) = L_{Secant}[(x - \delta, g(x - \delta), (x + \delta, g(x + \delta)] & ; \\ \text{with } m_{\parallel} \text{ then } L_{tan} := y = (x - x_g) \cdot m_{||} + g(x_g) & ; \\ L_{tan} \text{ tangent to the point } (x_g, g(x_g)) \end{cases}$$

$$ker(d_{\leftarrow}) = \{x \in \{E\}_{i=1}^{\infty} | f_{Cochoice}(x) = e_{(Im\left(L_{tan(x,f(x))}\right),\cdot)}\} \quad ;$$

$$Im(d_{\leftarrow}) := Im(L_{tan(x,f(x)} \subseteq \Re$$

i.e. the cohomology group is defined with integration (trapezoidal integration) over all N for a function $f(x)$.

with $(x, f(x)) \in \mathbb{C}$ used to define $(u, v) \in \mathbb{C}$ and $(fdz, d\bar{w})$ defining the wedge product $fdz \wedge d\bar{w}$

with

$$[fdz, d\bar{w}]_{d_{\leftarrow}} := \{ \quad (fdz, d\bar{w}) \quad | \quad z; w \text{ holomorphic for } d_{\leftarrow} N \text{ and } H_{d_{\leftarrow} N} \}$$

with the Hodge class and the ω - form defined by fdz and $d\bar{w}$ by wedge product and the algebraic variety where P_k is defined; if the variety contain functions hence

$$f_T = n_1 f_1 + h_2 f_2 + \cdots + n_n f_n$$

is defined (i.e. an algebraic cycle can be established)

$$-o-$$

19 Commentaries

Some definitions like that about bilinear form has been changed to establish the meaning derived from the properties of the algebraic structure used to establish such propositions.

This means that prior definition is assumed to be useless,; aim to establish relations among mathematical entities that cannot be developed; on the case of bilinear form the determinant of the matrix used to define the skew-symmetric pairing is not distributive.

20 References

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Chatgpt was used to review some definitions used on this work

Copilot Microsoft Bing was used on this work for same purpose